Transient Solution of $M^{[X]}$ / G /1 With Second Optional Service, Bernoulli Schedule Server Vacation and Random Break Downs

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ABSTRACT

In this model, we present a batch arrival non- Markovian queuing model with second optional service, subject to random break downs and Bernoulli vacation. Batches arrive in Poisson stream with mean arrival rate $\lambda(>0)$, such that all customers demand the first `essential' service, wherein only some of them demand the second `optional' service. The service times of the both first essential service and the second optional service are assumed to follow general (arbitrary) distribution with distribution function $B_1(v)$ and $B_2(v)$ respectively. The server may undergo breakdowns which occur according to Poisson process with breakdown rate α . Once the system encounter break downs it enters the repair process and the repair time is followed by exponential distribution with repair rate β . Also the sever may opt for a vacation according to Bernoulli schedule. The vacation time follows general (arbitrary)distribution with distribution function v(s). The time-dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been derived explicitly. Also the mean queue length and the mean waiting time have been found explicitly.

Keywords: $M^{[X]}$ /G/1 Queue, First essential service, Second optional service, Bernoulli schedule, Probability generating function, Transient state, Steady state.

SUBJECT CLASSIFICATION

AMS 60K25, 60K30

1. INTRODUCTION

The research study on queuing systems with server vacation has become an extensive and interesting area in queuing theory literature. Server vacations are used for utilization of idle time for other purposes. Vacation queuing models has been modelled effectively in various situations such as production, banking service, communication systems, and computer networks etc. Numerous authors are interested in studying queuing models with various vacation policies including single and multiple vacation policies. Batch arrival queue with server vacations was investigated by Yechiali(1975). An excellent comprehensive studies on vacation models can be found in Takagi(1991) and Doshi(1986) research papers. One of the classical vacation model in queuing literature is Bernoulli scheduled server vacation. Keilson and Servi(1987) introduced and studied vacation scheme with Bernoulli schedule discipline. Madan(2001) studied queue system with compulsory vacation in which the server should go for vacation with probability 1 whenever the system be comes empty. Later on, the same author discussed many queuing models with Bernoulli scheduled server vacation Baba(1986) employed the supplementary variable technique for deriving the transform solutions of waiting time for batch arrival with vacations.

Relating with the server of a queuing system, the server may be assumed as a reliable one, but this is not the case in most of real scenarios that the server will not be last for ever in order to provide service. So in this context, numerous papers of the server may be assumed unreliable, which can encounter breakdown. Thus queuing model with server break down is a remarkable and unavoidable phenomenon and the study of queues with server breakdowns and repairs has importance not only in the point of theoretic view but also in the engineering applications. Avi-Itzhak(1963) considered some queuing problems with the servers subject to breakdown. Kulkarni et al.(1990) studied retrial queues with server subject to breakdowns and repairs. Tang(1997) studied M/G/1 model with server break down and discussed reliability of the system. Madan et al.(2003) obtained the steady state results of single server Markovian model with batch service subject to queue models with random breakdowns. Queuing systems with random break downs and vacation have also been keenly analyzed by many authors including Grey (2000) studied vacation queuing model with service breakdowns. Madan and Maraghi (2009) have obtained steady state solution of batch arrival queuing system with random breakdowns and Bernoulli schedule server vacations having general vacation time. Thangaraj(2010) studied the transient behaviour of single server with compulsory vacation and random break downs.

Queuing models with Second optional service plays a prominent role in the research study of queuing theory. In this type of queuing model, the server performs first essential service to all arriving customers and after completing the first essential service, second optional service will be provided to some customers those who demand a second optional service. Madan(2000) has first introduced the concept of second optional service of an M/G/1 queuing system in which he has analyzed the time-dependent as well as the steady state behaviour of the model by using supplementary variable technique. Medhi(2001)proposed an M/G/1 queuing model with second optional channel who developed the explicit expressions for the mean queue length and mean waiting time. Later Madhan(2002) studied second optional service by incorporating Bernoulli schedule server vacations. Gaudham. Choudhury(2003) analyzed some aspects of M/G/1 queuing system with second optional service and obtained the steady state queue size distribution at the stationary point of time for general second optional service. A batch arrival with two phase service model with re-service for each phase of the service has been analyzed by Madan et al.(2004). Wang (2004) studied an M/G/1 queuing system with second optional service and server breakdowns based on supplementary variable technique. Kalyanaraman et al.(2008) studied additional optional batch service with vacation for single server queue.

In this paper we consider queuing system such that the customers are arriving in batches according to Poisson stream. The server provides a first essential service to all incoming customers and a second optional service will be provided to only some of them those who demand it. Both the essential and optional service times are assumed to follow general distribution. As soon as the service of a customer is complete, the server may go for a vacation with probability p or the server may continue to stay in system to serve the next customer with probability 1-p, if any. The vacation times are also assumed to be general whereas the repair time is exponentially distributed. Whenever the system meets a break down, it enters in to a repair process and the customer whose service is interrupted goes back to the head of the queue. Customers arrive in batches to the system and are served on a first come-first served basis.

The rest of the paper is organized as follows. The mathematical description of our model is in Section 2 and equations governing the model are given in Section 3. The time dependent solution have been obtained in Section 4 and the corresponding steady state results have been derived explicitly in Section 5.

2. MATHEMATICAL DESCRIPTION OF THE MODEL

The following assumptions are to be used describe the mathematical model of our study:

- Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided service one by one on a 'first come first served' basis. Let c_i dt (i = 1, 2, 3, . . .) be the first order probability that a batch of i customers arrives at the system during a short interval of time (t, t + dt], where $0 \le c_i \le 1$ and $\sum c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. Let $\lambda c_i \Delta t$; i = 1, 2, 3, ... be the first order probability of arrival of 'i' customers in batches in the system during a short period of time(t, t+dt) where $0 \le c_i \le 1$; $\sum_{i=1}^{n} c_i = 1$ $\lambda > 0$ is the mean arrival rate of batches.
- There is a single server which provides the first essential service to all arriving customers. Let $B_1(v)$ and $b_1(v)$ respectively be the distribution function and the density function of the first service times respectively.
- As soon as the first service of a customer is completed, then he may demand for the second service with probability r, or else he may decide to leave the system with probability 1- r in which case another customer at the head of the queue (if any) is taken up for his first essential service.
- The second service times as assumed to be general with the distribution function $B_2(v)$ and the density function $b_2(v)$. Further, Let $\mu_i(x)dx$ be the conditional probability density function of i^{th} service completion during the interval (x, x+dx] given that the elapsed service time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)} \tag{1}$$

and therefore

$$b_i(v) = \mu_i(v) e^{-\int_0^v \mu_i(x)dx} \tag{2}$$

- As soon as the customer's service is completed, the server may go for a vacation of random length V with probability p (0 < p < 1) or it may continue to serve the next customer with probability (1 - p).
- The vacation time of the server follows general (arbitrary) distribution with distribution function V(s) and the density function v(s). Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval (x, x+dx] given that the elapsed vacation time is x so that

$$\nu(x) = \frac{\gamma(x)}{1 - V(x)} \tag{3}$$

and therefore

$$\nu(s) = \gamma(s) e^{-\int_0^{vs} \gamma(x)dx} \tag{4}$$

- On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.
- The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$:

- Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate β > 0.
- Various stochastic processes involved in the system are independent of each others.

3. DEFINITIONS AND EQUATIONS GOVERNING THE SYSTEM

(i) $P_n^{(i)}(x, t)$ = probability that at time 't' the server is active providing i[th] service and there are 'n' $n \geq 1$ customers in the queue including the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(i)}(t)$ denotes the probability that at time 't' there are 'n' customers in the queue excluding the one customer in ith service irrespective of the value of x.

(ii) $V_n(x, t)$ = probability that at time 't', the server is on vacation with elapsed vacation time x, and there are 'n' $n \geq 1$ customers waiting in the queue for service. Consequently $V_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

(iii) $R_n(t)$=Probability that at time t, the server is inactive due to break down and the system is under repair while there are 'n' ($n \geq 0$) customers in the queue.

(iv) Q(t) = probability that at time 't' there are no customers in the system and the server is idle but available in the .

The queuing model is then, governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial t}P_n^{(1)}(x,t) + \frac{\partial}{\partial x}P_n^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha)P_n^{(1)}(x,t) = \lambda\sum_{i=1}^{n-1} c_i \; P_{n-i}^{(1)}(x,t) : n \geq 1 \tag{5}$$

$$\frac{\partial}{\partial t}P_0^{(1)}(x,t) + \frac{\partial}{\partial x}P_0^{(1)}(x,t) + (\lambda + \mu_1(x) + \alpha)P_0^{(1)}(x,t) = 0 \tag{6}$$

$$\frac{\partial}{\partial t}P_n^{(2)}(x,t) + \frac{\partial}{\partial x}P_n^{(2)}(x,t) + (\lambda + \mu_2(x) + \alpha)P_n^{(2)}(x,t) = \lambda\sum_{i=1}^{n-1} c_i \; P_{n-i}^{(2)}(x,t) \tag{7}$$

$$\frac{\partial}{\partial t}P_0^{(2)}(x,t) + \frac{\partial}{\partial x}P_0^{(2)}(x,t) + (\lambda + \mu_2(x) + \alpha)P_0^{(2)}(x,t) = 0 \tag{8}$$

$$\frac{\partial}{\partial t}V_n(x,t) + \frac{\partial}{\partial x}V_n(x,t) + (\lambda + \gamma(x))V_n(x,t) = \lambda\sum_{i=1}^{n-1} c_i V_{n-i}(x,t); n \geq 1 \tag{9}$$

$$\frac{\partial}{\partial t}V_0(x,t) + \frac{\partial}{\partial x}V_0(x,t) + (\lambda + \gamma(x))V_0(x,t) = 0 \tag{10}$$

$$\frac{d}{dt}R_n(t) = -(\lambda + \beta)R_n(t) + \lambda\sum_{i=1}^{n-1} c_i R_{n-i}(t) + \alpha\int_0^\infty P_{n-1}^{(1)}(x,t)dx + \alpha\int_0^\infty P_{n-1}^{(2)}(x,t)dx \tag{11}$$

$$\frac{d}{dt}R_0(t) = -(\lambda + \beta)R_0(t) \tag{12}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t) + \beta R_0(t) + (1-p)(1-r)\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx + (1-p)\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_0(x,t)\gamma(x)dx \tag{13}$$

Equations (5) to (13) are to be solved subject to the following boundary conditions.

$$P_0^{(1)}(0,t) = \lambda c_1 Q(t) + \beta R_1(t) + (1-p)(1-r)\int_0^\infty P_1^{(1)}(x,t)\mu_1(x)dx + (1-p)\int_0^\infty P_1^{(2)}(x,t)\mu_2(x)dx + \int_0^\infty V_1(x,t)\gamma(x)dx \tag{14}$$

$$P_n^{(1)}(0,t)=\lambda c_{n+1}Q(t)+\beta R_{n+1}(t)+(1-p)(1-r)\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx+(1-p)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx+\int_0^\infty V_{n+1}(x,t)\gamma(x)dx;\, n\geq 1 \quad (15)$$

$$P_n^{(2)}(0,t)=r\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx;\, n\geq 0 \quad (16)$$

$$V_n^{(1)}(0,t)=p(1-r)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx+p\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx;\, n\geq 0 \quad (17)$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

With Q (0) = 1, V_n (0)=0, R (0) = 0 and $P_n^{(i)}(0)$ = 0, i=1,2 ; n=0,1,2,… (18)

4. TIME DEPENDENT SOLUTION

Generating functions of the queue length

Now we define the probability generating function as follows

$$P_q^{(i)}(x,z,t)=\sum_{n=0}^{\infty}P_n^{(i)}(x,t)z^n;\, i=1,2 \quad P_q^{(i)}(z,t)=\sum_{n=0}^{\infty}P_n^{(i)}(t)z^n;\, i=1,2$$

$$V_q(x,z,t)=\sum_{n=0}^{\infty}V_n(x,t)z^n \quad V_q(z,t)=\sum_{n=0}^{\infty}V_n(t)z^n$$

$$R_q(z,t)=\sum_{n=0}^{\infty}R_n(t)z^n$$

$$C(z)=\sum_{n=1}^{\infty}c_n z^n$$

(19)

Which are convergent inside the circle given by │z│≤1 and define the Laplace transform of a function f(t) as

$$\bar{f}(s)=\int_0^\infty f(t)e^{-st}dt \quad (20)$$

Taking Laplace transforms of equations (5) to (17)

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\bar{P}_n^{(1)}(x,s)=\lambda\sum_{i=1}^{n-1}c_i\bar{P}_{n-i}^{(1)}(x,s);\, n\geq 1 \quad (21)$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\bar{P}_0^{(1)}(x,s)=0 \quad (22)$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s)+(s+\lambda+\mu_2(x)+\alpha)\bar{P}_n^{(2)}(x,s)=\lambda\sum_{i=1}^{n-1}c_i\bar{P}_{n-i}^{(2)}(x,s);\, n\geq 1 \quad (23)$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s)+(s+\lambda+\mu_2(x)+\alpha)\bar{P}_0^{(2)}(x,s)=0 \quad (24)$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s)+(s+\lambda+\gamma(x))\bar{V}_n(x,s)=\lambda\sum_{i=1}^{n=1}c_i\bar{V}_{n-i}(x,s);\, n\geq 0;\, n\geq 1 \quad (25)$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s)+(s+\lambda+\gamma(x))\bar{V}_0(x,s)=0$$

(26)

$$(s+\lambda+\beta)\bar{R}_n(s)=\lambda\sum_{i=1}^{n-1}c_i\bar{R}_{n-i}(t)+\alpha\int_0^\infty\bar{P}_{n-1}^{(1)}(x,s)dx+\alpha\int_0^\infty\bar{P}_{n-1}^{(2)}(x,s)dx \quad (27)$$

$$(s+\lambda+\beta)\bar{R}_0(s)=0 \tag{28}$$

$$(s+\lambda)\bar{Q}(s)=1+\beta\bar{R}_0(s)+(1-r)(1-p)\int_0^\infty \bar{P}_0^{(1)}(x,s)\mu_1(x)dx+(1-p)\int_0^\infty \bar{P}_0^{(2)}(x,s)\mu_2(x)dx+\int_0^\infty \bar{V}_0(x,s)\gamma(x)dx \tag{29}$$

$$\bar{P}_0^{(1)}(0,s)=\lambda c_1\bar{Q}(s)+\beta\bar{R}_1(s)+(1-r)(1-p)\int_0^\infty \bar{P}_1^{(1)}(x,s)\mu_1(x)dx+(1-p)\int_0^\infty \bar{P}_1^{(2)}(x,s)\mu_2(x)dx+\int_0^\infty \bar{V}_1(x,s)\gamma(x)dx \tag{30}$$

$$\bar{P}_n^{(1)}(0,s)=\lambda c_{n+1}\bar{Q}(s)+\beta\bar{R}_{n+1}(s)+(1-r)(1-p)\int_0^\infty \bar{P}_{n+1}^{(1)}(x,s)\mu_1(x)dx+(1-p)\int_0^\infty \bar{P}_{n+1}^{(2)}(x,s)\mu_2(x)dx+\int_0^\infty \bar{V}_{n+1}(x,s)\gamma(x)dx \tag{31}$$

$$\bar{P}_n^{(2)}(0,s)=r\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx;n=0,1,2,... \tag{32}$$

$$\bar{V}_n(0,s)=p(1-r)\int_0^\infty \bar{P}_n^{(1)}(x,t)\mu_1(x)dx+p\int_0^\infty \bar{P}_n^{(2)}(x,t)\mu_2(x)dx;n\ge 0 \tag{33}$$

Now multiplying equation (21) by z^n and summing over n from 1 to ∞, adding to equation (22) and using the definition of probability generating function, we obtain

$$\frac{\partial}{\partial x}\bar{P}_q^{(1)}(x,z,s)+(s+\lambda-\lambda C(z)+\mu_1(x)+\alpha)\bar{P}_q^{(1)}(x,z,s)=0 \tag{34}$$

Performing similar operations on equations (23) to (28)

$$\frac{\partial}{\partial x}\bar{P}_q^{(2)}(x,z,s)+(s+\lambda-\lambda C(z)+\mu_2(x)+\alpha)\bar{P}_q^{(2)}(x,z,s)=0 \tag{35}$$

$$\frac{\partial}{\partial x}\bar{V}_q(x,z,s)+(s+\lambda-\lambda C(z)+\gamma(x))\bar{V}_q(x,z,s)=0 \tag{36}$$

$$(s+\lambda-\lambda C(z)+\beta)\bar{R}_q(z,s)=\alpha z\int_0^\infty \bar{P}_q^{(1)}(x,z,s)dx+\alpha z\int_0^\infty \bar{P}_q^{(2)}(x,z,s)dx \tag{37}$$

For the boundary conditions, multiply both sides of equation (30) by z ,multiply both sides of equation (31) by z^{n+1}, summing over 1 to ∞, adding the two results and using the definition of probability generating function equation we get,

$$z\bar{P}_q^{(1)}(0,z,s)=(1-r)(1-p)\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx+(1-p)\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx+\int_0^\infty \bar{V}_q(x,z,s)\gamma(x)dx+\beta\bar{R}_q(z,s)+\lambda[C(z)-1]\bar{Q}(s)+[1-s\bar{Q}(s)] \tag{38}$$

Performing similar operation on equations (32) and (33) we obtain

$$\bar{P}_q^{(2)}(0,z,s)=r\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx \tag{39}$$

$$\bar{V}_q(0,z,s)=p(1-r)\int_0^\infty \bar{P}_q^{(1)}(x,z,t)\mu_1(x)dx+p\int_0^\infty \bar{P}_q^{(2)}(x,z,t)\mu_2(x)dx;n\ge 0 \tag{40}$$

Integrating the equation (34) from 0 to x yields

$$\bar{P}_q^{(1)}(x,z,s)=\bar{P}_q^{(1)}(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu_1(t)dt} \tag{41}$$

where $\bar{P}_q^{(1)}(0,z,s)$ is given by equation (38)

Again integrating equation (41) by parts with respect to x yields

$$\bar{P}_q^{(1)}(z,s) = \bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \tag{42}$$

Where

$$\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x} dB_1(x) \tag{43}$$

is Laplace - Stieltjes transform of the first essential service time $B_1(x)$. Now multiplying both sides of equation (41) by $\mu_1(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}_q^{(1)}(0,z,s)\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) \tag{44}$$

Similarly, on integrating equation (35) and (36) from 0 to x, we get

$$\bar{P}_q^{(2)}(x,z,s) = \bar{P}_q^{(2)}(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x \mu_2(t)dt} \tag{45}$$

$$\bar{V}_q(x,z,s) = \bar{V}_q(0,z,s)e^{-(s+\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \tag{46}$$

where $\bar{P}_q^{(2)}(0,z,s)$ and $V_q(0; z; s)$ are given by equations (39) and (40).

Again integrating equations (45) and (46) by parts with respect to x yields

$$\bar{P}_q^{(2)}(z,s) = \bar{P}_q^{(2)}(0,z,s)\left[\frac{1-\bar{B}_2(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \tag{47}$$

$$\bar{V}_q(z,s) = \bar{V}_q(0,z,s)\left[\frac{1-\bar{V}(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z)}\right] \tag{48}$$

Where

$$\bar{B}_2(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x} dB_2(x) \tag{49}$$

is Laplace - Stieltjes transform of the second optional service time $B_2(x)$ and

$$\bar{V}(s+\lambda\ (1-C(z))) = \int_0^\infty e^{-(s+\lambda-\lambda C(z))x} dV(x) \tag{50}$$

is Laplace - Stieltjes transform of the service time V(x).
Now multiplying both sides of equation (45) by $\mu_2(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx = \bar{P}_q^{(1)}(0,z,s)\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) \tag{51}$$

Now using equation (44), equation (39) reduces to

$$\bar{P}_q^{(2)}(0,z,s) = r\bar{P}_q^{(1)}(0,z,s)\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) \tag{52}$$

Now multiplying both sides of equation (46) b γ (x) and integrating over x, we get

$$\int_0^\infty \bar{V}_q(x,z,s)\gamma(x)dx = \bar{V}_q(0,z,s)\bar{V}(s+\lambda-\lambda C(z)) \tag{53}$$

the equation (47)becomes

$$\bar{P}_q^{(2)}(z,s) = r\,\bar{P}_q^{(1)}(0,z,s)\left[\frac{\bar{B}_1(s+\lambda-\lambda C(z)+\alpha)(1-\bar{B}_2(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \qquad (54)$$

Now using equations (44), (51) and (52), equation (40) can be written as

$$\bar{V}_q(0,z,s) = p(1-r)\bar{P}_q^{(1)}(0,z,s)\,\bar{B}_1(s+\lambda-\lambda C(z)+\alpha) + p\,r\bar{P}_q^{(1)}(0,z,s)\,\bar{B}_1(s+\lambda-\lambda C(z)+\alpha)(\bar{B}_2(s+\lambda-\lambda C(z)+\alpha) \qquad (55)$$

Using above equation (55), equation (48) becomes

$$\bar{V}(z,s) = \bar{P}_q^{(1)}(0,z,s)\,p\{(1-r)\bar{B}_1[f_1(z)] + r\,\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]\}\left[\frac{(1-\bar{V}\;(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z))}\right] \qquad (56)$$

Where

$$f_1(z) = s+\lambda-\lambda C(z)+\alpha$$

Using equations (42), (51) and (54) equation (37) becomes

$$\bar{R}_q(z,s) = \frac{\alpha z\,\bar{P}_q^{(1)}(0,z,s)[1-(1-r)\bar{B}_1[f_1(z)] - r\,\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]]}{[f_1(z)][f_2(z)]} \qquad (57)$$

Where

$$f_2(z) = s+\lambda-\lambda C(z)+\beta$$

Now using equations (44), (51), (53) and (56) in equation (38) and solving for, $\bar{P}_q^{(1)}(0,z,s)$, we get

$$\bar{P}_q^{(1)}(0,z,s) = \frac{[f_1(z)][f_2(z)][\lambda(C(z)-1)\bar{Q}(s)+(1-s\,\bar{Q}(s))]}{Dr} \qquad (58)$$

where

$$Dr = f_1(z)f_2(z)\{z-[(1-p)-p\bar{V}(s+\lambda-\lambda C(z))][(1-r)\bar{B}_1[f_1(z)]+r\,\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]\}-\alpha\beta z[1-(1-r)\bar{B}_1[f_1(z)]-r\,\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]] \qquad (59)$$

Substituting the value of $\bar{P}_q^{(1)}(0,z,s)$ from equation (57) in to equation (42), (47), (55) and (56) we get

$$\bar{P}_q^{(1)}(z,s) = \frac{f_2(z)[1-\bar{B}_1[f_1(z)]][\lambda(C(z)-1)\bar{Q}(s)+(1-s\,\bar{Q}(s))]}{Dr} \qquad (60)$$

$$\bar{P}_q^{(2)}(z,s) = \frac{rf_2(z)\bar{B}_1[f_1(z)][1-\bar{B}_2[f_1(z)]][\lambda(C(z)-1)\bar{Q}(s)+(1-s\,\bar{Q}(s))]}{Dr} \qquad (61)$$

$$\bar{V}_q(z,s) = \frac{pf_1(z)f_2(z)[(1-r)\bar{B}_1[f_1(z)]+r\,\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]][\lambda(C(z)-1)\bar{Q}(s)+(1-s\,\bar{Q}(s))]}{Dr}\left[\frac{(1-\bar{V}\;(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z))}\right] \qquad (62)$$

$$\bar{R}_q(z,s) = \frac{\alpha z[1-(1-r)\bar{B}_1[f_1(z)]-r\,\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]][\lambda(C(z)-1)\bar{Q}(s)+(1-s\,\bar{Q}(s))]}{Dr} \qquad (63)$$

Where Dr is given by equation (58)

5. THE STEADY STATE ANALYSIS

In this section derive the steady state probability distribution for our queuing model. To define the steady state probabilities, suppress. The argument 't' wherever it appears in the time dependent analysis.
By using well known Tauberian property

$$Lt_{s\to 0}\,s\,\bar{f}(s) = Lt_{t\to\infty}\,f(t) \qquad (64)$$

multiplying both sides of equation (60), (61), (62) and (63) by s and applying property (64) and simplifying, we get

$$P_q^{(1)}(z) = \frac{f_2(z)[1-B_1[\bar{f}_1(z)]][\lambda(C(z)-1)Q]}{Dr} \quad (65)$$

$$P_q^{(2)}(z) = \frac{rf_2(z)\bar{B}_1[f_1(z)][1-B_2[\bar{f}_1(z)]][\lambda(C(z)-1)Q]}{Dr}$$

(66)

$$V_q(z) = \frac{pf_1(z)f_2(z)[(1-r)B_1[f_1(z)]+r\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]][\bar{V}(\lambda-\lambda C(z))-1]Q]}{Dr} \quad (67)$$

$$R_q(z) = \frac{\alpha z[1-(1-r)B_1[f_1(z)]-r\bar{B}_1[f_1(z)]\bar{B}_2[f_1(z)]][\lambda(C(z)-1)Q]}{Dr} \quad (68)$$

Let $W_q(z)$ denotes the probability generating function of queue size irrespective of the state of the system.

Then adding (65), (66), (67) and (68) we get

$$W_q(z) = P_q^{(1)}(z) + P_q^{(2)}(z) + V_q(z) + R_q(z) \quad (69)$$

In order to obtain Q, using the normalization condition,

$$W_q(1) + Q = 1 \quad (70)$$

We see that for z=1, $W_q(z)$ is indeterminate of the form 0/0. We apply L' Hospital's rule in equation (69), where $B_i(0) = 1$; i = 1; 2; _ V (0) = 1;$V_0(0)$ = E[V] the mean vacation time.

$$P_q^{(1)}(1) = \frac{\lambda\beta QE(I)[1-\bar{B}_1(\alpha)]}{dr} \quad (71)$$

$$P_q^{(2)}(1) = \frac{r\lambda\beta QE(I)[1-\bar{B}_2(\alpha)]\bar{B}_1(\alpha)}{dr} \quad (72)$$

$$V_q(1) = \frac{p\alpha\beta\lambda QE(I)E(I)[(1-r)B_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]}{dr} \quad (73)$$

$$R_q(1) = \frac{\alpha\lambda QE(I)[1-(1-r)B_1(\alpha)-r\bar{B}_1(\alpha)\bar{B}_2(\alpha)]}{dr} \quad (74)$$

Where dr is given by

$$dr = \alpha\beta[(1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] - \lambda E(I)(\alpha+\beta)[1-(1-r)\bar{B}_1(\alpha)-r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] - \lambda\alpha\beta pE(I)E(V)[(1-r)\bar{B}_1(\alpha)-r\bar{B}_1(\alpha)\bar{B}_2(\alpha)] \quad (75)$$

$$Q = 1-\lambda E(I)\left[\frac{1}{\beta(1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)} + \frac{1}{\alpha(1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)} - \frac{1}{\alpha} - \frac{1}{\beta} + pE(V)\right] \quad (76)$$

and the utilization factor ρ of the system is given by

$$\rho = \lambda E(I)\left[\frac{1}{\beta(1-r)\bar{B}_1(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)} + \frac{1}{\alpha(1-r)\bar{B}(\alpha)+r\bar{B}_1(\alpha)\bar{B}_2(\alpha)} - \frac{1}{\alpha} - \frac{1}{\beta} + pE(V)\right] \quad (77)$$

Where ρ < 1 is the stability condition under which the steady state exists, equation (76) gives the probability that the server is idle. Substitute Q from equation (76) in equation (69), $W_q(z)$ have been completely and explicitly determined which is the probability generating function of the queue size.

The Average Queue Size

Let L_q denote the mean number of customers in the queue under the steady state, then $L_q = d/dz\, [W_q(z)]_{z=1}$ .since this formula gives 0/0 form, then we write $W_q(z) = N(z) / D(z)$ where N(z) and D(z) are the numerator and denominator of the right hand side of equation (69) respectively, then we use

$$L_q = \frac{D'N'' - D''N'}{2(D')^2} \qquad (78)$$

where primes and double primes in equation (78) denote first and second derivation at z=1 respectively. Carrying out the derivatives at z=1 , we have

$$N'(1) = Q\lambda E(I)\left[(\alpha+\beta) + ((1-r)\bar{B}_1(\alpha) + r\,\bar{B}_1(\alpha)\,\bar{B}_2(\alpha))[p\alpha\beta E(V) - \alpha - \beta]\right] \qquad (79)$$

$$N''(1) = 2Q[\lambda E(I)]^2 \left[\begin{array}{l} (\frac{\alpha}{\lambda E(I)} - 1) + ((1-r)\bar{B}_1(\alpha) + r\,\bar{B}_1(\alpha)\,\bar{B}_2(\alpha))[1 - \frac{\alpha}{\lambda E(I)} - p(\alpha+\beta)E(V) + \frac{1}{2}p\alpha\beta E(V^2)] \\ + ((1-r)\bar{B}_1{}'(\alpha) + r(\bar{B}_1{}'(\alpha) + \bar{B}_2{}'(\alpha))[\alpha+\beta - p\alpha\beta E(V)] \end{array}\right]$$

$$+ \lambda E(I(I-1))Q[(\alpha+\beta) + ([(1-r)\bar{B}_1(\alpha) + r\,\bar{B}_1(\alpha)\,\bar{B}_2(\alpha))[p\alpha\beta E(V) - \alpha - \beta] \qquad (80)$$

$$D'(1) = -\lambda E(I)(\alpha+\beta) + \left[((1-r)\bar{B}_1(\alpha) + r\,\bar{B}_1(\alpha)\,\bar{B}_2(\alpha))[\alpha\beta + \lambda E(I)((\alpha+\beta) - p\alpha\beta E(V))]\right] \qquad (81)$$

$$D''(1) = 2[\lambda E(I)]^2 \left[\begin{array}{l} (1 - \frac{\alpha+\beta}{\lambda E(I)}) + ((1-r)\bar{B}_1(\alpha) + r\,\bar{B}_1(\alpha)\,\bar{B}_2(\alpha))[-1 + p(\alpha+\beta)E(V) - \frac{1}{2}p\alpha\beta E(V^2)] \\ + ((1-r)\ \bar{B}_1{}'(\alpha) + r(\bar{B}_1{}'(\alpha) + \bar{B}_2{}'(\alpha))[-(\alpha+\beta) - \frac{\alpha\beta}{\lambda E(I)} + p\alpha\beta E(V)] \end{array}\right]$$

$$+ \lambda E(I(I-1))[-(\alpha+\beta) + ([(1-r)\bar{B}_1(\alpha) + r\,\bar{B}_1(\alpha)\,\bar{B}_2(\alpha))[\alpha+\beta - p\alpha\beta E(V)] \qquad (82)$$

where E(V 2) is the second moment of the vacation time and Q has been found in equation (76). Then if we substitute the values of N'(1), N"(1), D'(1) and D"(1) from equations (79), (80), (81) and (82) in to (78) equation we obtain L_q in a closed form. Mean waiting time of a customer could be found as

$$W_q = \frac{L_q}{\lambda} \qquad (83)$$

by using Little's formula.

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