DOI: https://doi.org/10.24297/ijct.v23i.9363

Viscous Stability Criterion for Hydrodynamic Differential Rotation

Zinab M. Maatoug [1], Hana N. Albibas[1], Bashir W. Sharif [2], Ali M. Awin [2]

[1]Department of Mathematics, Faculty of Education, University of Tripoli- Genzor, Libya

[2]Department of Mathematics, Faculty of Science, University of Tripoli, Libya

Email: awinsus@yahoo.com

Abstract

Viscous stability criterion in a thin layer on a rotating sphere is studied. The case when the fluid is inviscid was explained by Watson in 1981, the work was motivated by the idea suggested by Drazin and Ried in their celebrated text "Hydrodynamic Stability"; here we will investigate the model worked out by B. Sherif and C. Jones in the year 2005, and show the necessary condition for instability which depends on the energy that is provided by the shear motion of the fluid in spherical thin layer.

Keywords: fluid, wave number, phase velocity, viscous, inviscid, energy.

1. Introduction

The primary goal of this paper is to obtain a stability criterion for the model that was studied before [8]. In their work, they did set the equation for their model but did not explain the viscous stability criterion when a disturbance is present in the fluid, the energy contained within the disturbance must burst from the mean motion itself. To study the mechanism of this transfer of energy between the mean motion and the disturbance one is important in fluid mechanics. We will follow the same procedure adopted by Drazin and Reid [2]; the steps are given by Synge [10]. His results were improved by Joseph [4]. The Orr-Sommerfeld equation for our model is the essential equation for obtaining the sufficient condition for stability [7 & 9]. The process will introduce an energy equation for two-dimensional disturbances propagating in the direction of the basic flow. Hence in the next section we give the governing equations of motion of the fluid, then in section 3 we state the boundary conditions related to our problem. In section 4 we present the viscous stability criterion and finally in section 5, we conclude with a brief discussion.

2. Governing Equations

We start, first, with the perturbation equation that has been introduced earlier [8] which is

$$\left(\frac{\partial}{\partial t}+\ \Omega\frac{\partial}{\partial\emptyset}\right)\left(\nabla_H\right)^2\psi-\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(\ \Omega\, sin^2\theta\right)\right]\frac{\partial\psi}{\partial\emptyset}=R_e^{\ -1}\left(\nabla_H\right)^4\psi. \qquad (1)$$

Note that when the right-hand side of the above equation is equal to zero, we get the equation investigated by Watson [11], and where he obtained the onset of instability of the differential rotation for his model. Our task then, is to find the energy equation for our model and therefore to set the range for the point of instability for the wave that carried the energy.

We shall adapt the normal mode method to find the solution of Equation (1), and therefore to obtain the position of growing or decaying waves as time progresses after an initial and infinitesimal disturbance. Note that the above differential equation has coefficients depending on θ and $\left(\nabla_H\right)^2$, and therefore the solution will have an exponential form which is a function of $(\emptyset, t)$. In other words we look for a solution of the form

$$\psi(\emptyset,\theta\,,t)=\ \tilde{\psi}(\theta)e^{-im(\emptyset-wt)} \qquad (2)$$

where $\tilde{\psi}(\theta)$ is the amplitude of ψ and m is an integer which represents the longitudinal wave number. The phase velocity of the mode is given by $w\ =w_r+\ iw_i$. Now if we let $\mu\ =\ cos\,cos\,\theta$ as a new variable representing a new co-latitude and when substituting Equation (2) into Equation (1), we get a simple ordinary differential equation for; $\tilde{\psi}(\mu)$ namely, we get

$$(im\,R_e)^{-1}L^2\,\tilde{\psi} - (\,\Omega - w\,)\,\tilde{\psi} + \frac{d^2}{d\mu^2}\left[\Omega(\,1 - \mu^2\,)\right]\tilde{\psi} = 0, \qquad (3)$$

Where the operator

$$L = \frac{d}{d\mu}(\,1 - \mu^2\,)\frac{d}{d\mu} - \frac{m^2}{1-\mu^2}\,. \qquad (4)$$

3. Boundary Conditions

As one can note that the Equation (3) has regular singular point at $\mu = \pm\,1$. Since the equation is of the fourth order, we must exclude two singular solutions at $\mu = \pm\,1$.These conditions are implemented by expanding the solution in terms of associated Legendre functions $P_n^m(\mu)$ which are neglected at $\mu = \pm\,1$. Hence, our boundary condition for our model is

$\tilde{\psi}(\mu)$ is finite at $\mu = \pm\,1$ (5)

Equation (3), with the boundary condition in Equation (5), is called Orr-Sommerfeld equation. This equation is to be solved for given differential rotation profile $\Omega(\cos cos\ \cos cos\,\theta\ )$; R_e is Reynolds number and m is the wave number.

4. Viscous Stability Criterion

To obtain the viscous stability criterion for our model we refer the method outlined in the hydrodynamic stability textbook by Drazin and Reid [2]. First, multiplying Equation (1) by the conjugate of $\tilde{}$, which is $\tilde{\psi}^*$; then taking the integral over $\mu\epsilon[-\,1, 1]$. We get

$$\frac{-i}{mR_e}\int_{-1}^{1}\tilde{\psi}^*L^2(\tilde{\psi})d\mu = \int_{-1}^{1}(\Omega - w)\tilde{\psi}^*L(\tilde{\psi})d\mu - \int_{-1}^{1}|\ \ |^2\frac{d^2}{d\mu^2}\left[\Omega\left(1 - \mu^2\right)\right]d\mu\,. \qquad (6)$$

Now, $\int_{-1}^{1}\tilde{\psi}^*L^2(\tilde{\psi})d\mu$ is purely real. This because

$$\int_{-1}^{1}\tilde{\psi}^*L^2(\tilde{\psi})d\mu = \int_{-1}^{1}\tilde{\psi}^*\left(\frac{d}{d\mu}(1 - \mu^2)\frac{d}{d\mu} - \frac{m^2}{1-\mu^2}\right)L(\tilde{\psi})d\mu \quad = -\int_{-1}^{1}\frac{d}{d\mu}(1 - \mu^2)\frac{d\tilde{\psi}^*}{d\mu}L(\mu)d\,\mu - \int_{-1}^{1}\frac{m^2\tilde{\psi}^*}{1-\mu^2}L(\tilde{\psi})d\mu$$

$$= \int_{-1}^{1}L\left(\tilde{\psi}^*\right)L(\tilde{\psi})d\mu,$$

Hence, we get

$$\int_{-1}^{1}\tilde{\psi}^*\,L^2\,(\tilde{\psi})d\mu = \int_{-1}^{1}\left|L(\tilde{\psi})\right|^2d\mu \qquad (7)$$

As we claimed. Note that we used the integration by parts and the vanishing of $1 - \mu^2$ at the boundaries. Therefore, Equation (6) becomes

$$\frac{-i}{mR_e}\int_{-1}^{1}\left|L(\tilde{\psi})\right|^2d\mu = \int_{-1}^{1}(\Omega - w)\tilde{\psi}^*\,L(\tilde{\psi})d\mu - \int_{-1}^{1}|\ \ |^2\frac{d^2}{d\mu^2}\left[\Omega(1 - \mu^2)\right]d\mu. \qquad (8)$$

Since the second term of the right-hand side of Equation (8) is real, the imaginary part of this equation is then

$$\frac{-1}{mR_e}\int_{-1}^{1}\left|L(\tilde{\psi})\right|^2d\mu = Im\left\{\int_{-1}^{1}(\Omega - w)\tilde{\psi}^*L(\tilde{\psi})d\mu\right\} \qquad (9)$$

Now consider the integral

$$\int_{-1}^{1}(\Omega - w)\tilde{\psi}^*\,L(\tilde{\psi})d\mu$$

which is the same as

$$\int_{-1}^{1}(\Omega - w)\tilde{\psi}^{*}\left[\frac{d}{d\mu}\left(1-\mu^{2}\right)\frac{d\tilde{\psi}(\mu)}{d\mu} - \frac{m^{2}\tilde{\psi}(\mu)}{1-\mu^{2}}\right]d\mu = -\int_{-1}^{1}\frac{d}{d\mu}\left[(\Omega - w)\tilde{\psi}^{*}\right]\left(1-\mu^{2}\right)\frac{d\tilde{\psi}}{d\mu}d\mu$$

$$-\int_{-1}^{1}(\Omega - w)\frac{m^{2}\left|\tilde{\psi}(\mu)\right|^{2}}{1-\mu^{2}}d\mu \qquad (10)$$

Or as

$$\int_{-1}^{1}(\Omega - w)\tilde{\psi}^{*}L\left(\tilde{\psi}\right)d\mu = -\int_{-1}^{1}(\Omega - w)\left[\left(1-\mu^{2}\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^{2} + \frac{m^{2}\left|\tilde{\psi}(\mu)\right|^{2}}{1-\mu^{2}}\right]d\mu$$

$$-\int_{-1}^{1}\frac{d\Omega}{d\mu}\tilde{\psi}^{*}\left(1-\mu^{2}\right)\frac{d\tilde{\psi}(\mu)}{d\mu}d\mu. \qquad (11)$$

Again, we have used integration by parts and the vanishing terms at $\mu = \pm 1$ in evaluating this integration. Now the imaginary part of the R.H.S. of Equation (11) is simply

$$w_{i}\int_{-1}^{1}\left[\left(1-\mu^{2}\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^{2} + \frac{m^{2}\left|\tilde{\psi}(\mu)\right|^{2}}{1-\mu^{2}}\right]d\mu - Im\left\{\int_{-1}^{1}\frac{d\Omega}{d\mu}\tilde{\psi}^{*}\left(1-\mu^{2}\right)\frac{d\tilde{\psi}(\mu)}{d\mu}d\mu\right\}. \qquad (12)$$

From Equation (9) and Equation (12) we obtain

$$w_{i} = \frac{Im\left\{\int_{-1}^{1}\left(1-\mu^{2}\right)\frac{d\Omega}{d\mu}\tilde{\psi}^{*}\frac{d\tilde{\psi}(\mu)}{d\mu}d\mu\right\} - \frac{1}{mR_{e}}\int_{-1}^{1}\left|L(\tilde{\psi})\right|^{2}d\mu}{\int_{-1}^{1}\left[\left(1-\mu^{2}\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^{2} + \frac{m^{2}\left|\tilde{\psi}(\mu)\right|^{2}}{1-\mu^{2}}\right]d\mu}. \qquad (13)$$

We can see that the denominator is always positive and therefore for R_e to be negative (to indicate stability) we need to have

$$\frac{1}{mR_{e}}\int_{-1}^{1}\left|L\left(\tilde{\psi}\right)\right|^{2}d\mu > Im\left\{\int_{-1}^{1}\left(1-\mu^{2}\right)\frac{d\Omega}{d\mu}\tilde{\psi}^{*}\frac{d\tilde{\psi}(\mu)}{d\mu}d\mu\right\}. \qquad (14)$$

For small R_e, it is clear that Equation (14) will be satisfied and so the system must be stable. Physically, this means that if the system is very viscous, disturbances must decay away. We now seek to establish a definite value of $R_e = R_{crit}$ such that we can prove that the system is stable for all $R_e < R_{crit}$. We Know, from complex analysis, that $Im\,(z) = \frac{(z-z^{*})}{2i}$, then the R.H.S. of the inequality becomes

$$\frac{1}{2i}\int_{-1}^{1}\left(1-\mu^{2}\right)\frac{d\Omega}{d\mu}\left(\tilde{\psi}^{*}\frac{d\tilde{\psi}(\mu)}{d\mu} - \tilde{\psi}(\mu)\frac{d\tilde{\psi}^{*}}{d\mu}\right)d\mu. \qquad (15)$$

Taking the absolute value of this quantity

$$\left|\frac{1}{2i}\int_{-1}^{1}\left(1-\mu^{2}\right)\frac{d\Omega}{d\mu}\left(\tilde{\psi}^{*}\frac{d\tilde{\psi}(\mu)}{d\mu} - \tilde{\psi}(\mu)\frac{d\tilde{\psi}^{*}}{d\mu}\right)d\mu\right| \le \left[\left(1-\mu^{2}\right)^{\frac{1}{2}}\frac{d\Omega}{d\mu}\right]\int_{-1}^{1}\left|\tilde{\psi}(\mu)\right|\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\left(1-\mu^{2}\right)^{\frac{1}{2}}\right|d\mu. \qquad (16)$$

If we put

$$f = \left|\tilde{\psi}\right|, \qquad g = \left|\frac{d\tilde{\psi}(\mu)}{d\mu}\left(1-\mu^{2}\right)^{\frac{1}{2}}\right|,$$

and by Schwartz inequality,

$$\left(\int_{-1}^{1}f.g\,d\mu\right)^{2} \le \int_{-1}^{1}f^{2}\,d\mu\,.\int_{-1}^{1}g^{2}\,d\mu,$$

Then Equation (16) becomes

$$Im\left\{\int_{-1}^{1}\left(1-\mu^2\right)\frac{d\Omega}{d\mu}\tilde{\psi}^{*}\frac{d\tilde{\psi}(\mu)}{d\mu}d\mu\right\}\leq\left[\left(1-\mu^2\right)^{\frac{1}{2}}\frac{d\Omega}{d\mu}\right]\left\{\int_{-1}^{1}\left|\tilde{\psi}(\mu)\right|^2 d\mu\right\}^{\frac{1}{2}}\left\{\int_{-1}^{1}\left(1-\mu^2\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^2 d\mu\right\}^{\frac{1}{2}}. \quad (17)$$

Since the first and second terms on the R.H.S. of the above inequality are positive, then we can rewrite Equation (17) as

$$Im\left\{\int_{-1}^{1}\left(1-\mu^2\right)\frac{d\Omega}{d\mu}\tilde{\psi}^{*}\frac{d\tilde{\psi}(\mu)}{d\mu}d\mu\right\}\leq q\left\{\int_{-1}^{1}\left|\tilde{\psi}(\mu)\right|^2 d\mu\right\}^{\frac{1}{2}}\left\{\int_{-1}^{1}\left[\left(1-\mu^2\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^2+\frac{m^2\left|\tilde{\psi}(\mu)\right|^2}{1-\mu^2}\right]d\mu\right\}^{\frac{1}{2}}, \quad (18)$$

Where,

$$q=\left[\left(1-\mu^2\right)^{\frac{1}{2}}\frac{d\Omega}{d\mu}\right].$$

Note that

$$0<\int_{-1}^{1}\left(1-\mu^2\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^2\leq\int_{-1}^{1}\left[\left(1-\mu^2\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^2+\frac{m^2\left|\tilde{\psi}(\mu)\right|^2}{1-\mu^2}\right]d\mu$$

Since the second term on the R.H.S. of this inequality is positive. Hence, the R.H.S. of the inequality (18) informs us with the upper bound for w_i. What remains to show is the condition under which

$$\frac{1}{m R_e}\int_{-1}^{1}\left|L\left(\tilde{\psi}\right)\right|^2 d\mu\geq$$

$$q\left\{\int_{-1}^{1}\left|\tilde{\psi}(\mu)\right|^2 d\mu\right\}^{\frac{1}{2}}\left\{\int_{-1}^{1}\left[\left(1-\mu^2\right)\left|\frac{d\tilde{\psi}(\mu)}{d\mu}\right|^2+\frac{m^2\left|\tilde{\psi}(\mu)\right|^2}{1-\mu^2}\right]d\mu\right\}^{\frac{1}{2}} \quad (19)$$

holds.

If we let

$$\tilde{\psi}(\mu)=\sum_{n=m}^{\infty}a_n P_n^m(\mu), \quad L\left(\tilde{\psi}\right)=\sum_{n=m}^{\infty}-n(n+1)a_n P_n^m(\mu),$$

and by using the property of orthogonality

$$\int_{-1}^{1}P_n^m P_l^m d\mu=0 \quad if \quad n\neq l,$$

Then the inequality (19) becomes

$$\left\{\sum_{n=m}^{\infty}\alpha_n n^2(n+1)^2\right\}^2\geq m^2 R_e^2 q^2\left\{\sum_{n=m}^{\infty}\alpha_n\right\}\left\{\sum_{n=m}^{\infty}n(n+1)\alpha_n\right\}, \quad (20)$$

Where,

$$\alpha_n=\left|a_n\right|^2\int_{-1}^{1}\left(P_n^m\right)^2 d\mu.$$

α_n is either zero or positive. Furthermore, if we write the inequality (20) as

$$\left\{\sum_{n=m}^{\infty}\alpha_n n^2(n+1)^2\right\}^2\geq\frac{m^2 R_e^2 q^2}{m^3(m+1)^3}\left\{\sum_{n=m}^{\infty}m^2(m+1)^2\alpha_n\right\}\left\{\sum_{n=m}^{\infty}mn(m+1)(n+1)\alpha_n\right\}, \quad (21)$$

then the inequality still holds because all the terms in the sum are positive or zero provided that

$$\frac{R_e^2 q^2}{m(m+1)^3}<1.$$

Now note that since

$$n^2(n+1)^2 \geq m^2(m+1)^2 \text{ and } n^2(n+1)^2 \geq mn(m+1)(n+1),$$

then

$$\sum_{n=m}^{\infty} \alpha_n n^2(n+1)^2 > \sum_{n=m}^{\infty} \alpha_n m^2(m+1)^2$$

and

$$\sum_{n=m}^{\infty} \alpha_n n^2(n+1)^2 > \sum_{n=m}^{\infty} \alpha_n mn(m+1)(n+1)$$

respectively. So stability is guaranteed if

$$R_e < \frac{m^{\frac{1}{2}}(m+1)^{\frac{3}{2}}}{q}, \qquad (22)$$

Where,

$$q = \left[\left(1-\mu^2\right)^{\frac{1}{2}} \frac{d\Omega}{d\mu}\right].$$

This establishes the low Reynolds number bound. For R_e satisfying Equation (22) the flow must be stable, so all the plots of growth rate as a function of Reynolds number must have the property that the growth rate is negative for R_e less than the value in the inequality (22). Note that higher-m modes become stable because of the leading order of m.

It is to be noted that viscous stability is still of importance in many areas; for instance, in biology where asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system is of importance [6].

Conclusions

The stability criterion for hydrodynamic differential rotation of viscous flow in our model requires low Reynolds number. This is due to the energy driven by the interior waves to move the fluid particles after the initial disturbance.

The critical point for stability in our model is when $R_e < \frac{m^{\frac{1}{2}}(m+1)^{\frac{3}{2}}}{Q}$. The steps that we have used to obtain the critical point for stability are given in Reference [2]. Moreover, viscous stability keeps being an important subject till recently; for instance, in getting asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology [6] or in the study of viscous stability and instability when polymers are involved [5]. Moreover, an applicable method is introduced recently to verify global stability of a laminar fluid flow against all perturbations [3]; another recent work is on the electro-hydrodynamics stability [1].

References

[1]. Alali, E., Moatimid, G.M., Amer, M.F.E. (2022) EHD stability of two horizontal finite conducting rotating viscous fluids: Effects of energy and concentrations distributions, Results in Physics Vol. 40,105850. https://doi.org/10.1016/j.rinp.2022.105850

[2]. Drazin, P.G. Reid, W.H. (1981) Hydrodynamic stability.1st Edition, Cambridge University Press, Cambridge.

[3]. Fuentes, F., Goluskin, D., Chronyshenko, S. (2022) Global stability of fluid flows despite transient growth of energy, Rev. Lett. 128, 204502

https://doi.org/10.1103/PhysRevLett.128.204502

[4]. Joseph, D.D. (1968) Eigenvalue bounds for the Orr-Sommerfeld equation, Journal of Fluid Mechanics, Vol. 33 (3), pp. 617 - 621.

https://doi.org/10.1017/S0022112068001552

[5]. Liu, J., Song, W., Ma, G., Li, K. (2022) Faraday instability in viscous fluids covered with elastic polymers films, Polymers, vol. 14(12), 2334, pp. 1-15.

https://doi.org/10.3390/polym14122334.

[6]. Martinez, V.R., Wang, Z., Zhao, K. (2018) Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology, Indiana University Mathematics Journal, vol. 67 (4), pp. 1383-1424.

https://www.jstor.org/stable/45010333

[7]. Orr, W. (1907) The stability or instability of the steady motion of perfect liquid and viscous liquid. Proceeding of Royal Irish Academy, Vol. 27 (1907 - 1909), pp. 69-138.

https://www.jstor.org/stable/20490591

[8]. Sharif, B.W., Jones, C.A. (2005) Rotational and magnetic instability in the diffusive tachocline, Geophysical & Astrophysical Fluid Dynamics, vol. 99 (6), pp. 493-511.

https://doi.org/10.1080/03091920500372084.

[9]. Sommerfeld, A. (1908) Ein Beitrag zur hydrodynamischen erklaerung der turbuten fluessigkeitsbewegungen. Proceedings of 4th International Congress of Mathematicians, Rome, Vol. 3, pp. 116-124.

[10].Synge, J.L. (1938) Hydrodynamical stability, Semicentennial of the American Mathematical Society, vol 2, pp. 227-69, pp. 94,161,214,247,320.

https://www.ams.org/publicoutreach/math-history/procsemi-synge.pdf

[11]. Watson, M. (1981) Shear instability of differential rotation in stars. Geophysical and Astrophysical Fluid Dynamics, vol. 16 (1), pp. 285-298.

https://doi.org/10.1080/03091928008243663