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A Difference Scheme Using Spline for a Class of Singular Boundary Value Problem

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ABSTRACT

In this paper, a finite difference scheme using cubic splines has been discussed to find the numerical solution for a class of singular two point boundary value problems for certain ordinary differential equations. The cubic spline approximation leads to the tridiagonal system of equations, which can be solved using Newton's method. Second order convergence of the method has been established for quite general conditions. Two numerical examples are given to demonstrate the method and verify the second order accuracy.

Keywords

Splines, Singular Boundary Value Problem, Non-Uniform mesh.

1 INTRODUCTION

Mathematical research on the cubic spline functions and their application to the numerical solution of the differential equations has developed rapidly in the last few decades. The main advantages of using a cubic spline collocation procedure are that the governing matrix system obtained is always tridiagonal and the requirement of a uniform mesh is not necessary. It appears that the cubic spline approximation possesses some of the advantages of finite element techniques without the disadvantages of high computing cost and complex problem formulation.

Consider the class of singular two-point boundary value problems

$$(p(x)y')' = p(x)f(x,y), \qquad 0 < x \le 1, \tag{1}$$

$$y(0)=A, \quad \alpha y(1) + \beta\, y'(1) = \gamma, \tag{2}$$

where $\alpha > 0$, $\beta \ge 0$ and A, γ finite constants. We assume that $p(x)$ satisfies the following conditions

(**A**) (i) $p(x) > 0$ on $(0,1]$,

(ii) $p(x) \in C^1(0,1]$,

(iii) $p(x) = x^b g(x)$ on $[0,1]$, $b > 0$ and for some $r > 1$, $G(x) = 1/g(x)$ is analytic in $\{z : |z| < r\}$.

Further we assume

(**B**) for $(x,y) \in \{[0,1] \times \mathbb{R}\}$, $f(x,y)$ is continuous, $\partial f/\partial y$ exists, it is continuous and $\partial f/\partial y > 0$.

Existence-uniqueness of the problem (1) for general non-negative function $p(x)$ satisfying conditions (**A**)(i)–(iii) with boundary conditions $y(0) = A$ and $y(1) = B$ has been established in Ref. [8]. Existence-uniqueness for more general problem has been established in Ref. [6] with non-linear boundary condition at $x = 1$.

Singular two-point boundary value problems have been considered by various authors. Some second order Ref. [1, 2, 5, 7, 9, 11-13] as well as fourth-order Ref. [1, 3, 4, 10] convergent methods have been developed. Most of the authors have developed methods for the function $p(x) = x^b$, $0 \le b < 1$ and the boundary conditions $y(0) = A$ and $y(1) = B$.

In this article, second order spline method described in Ref. [7] are further extended to a class of non-negative functions $p(x)$ satisfying conditions (**A**)(i)–(iii) with boundary conditions (2). The order of accuracy of the method has been established for general class of functions $p(x)$ and under quite general conditions on $f(x,y)$. Numerical examples for general functions $p(x)$ are given to illustrate the method and verify the order of accuracy.

2 DESCRIPION OF THE SPLINE METHOD

For a positive integer $N \ge 2$, we consider a general non-uniform mesh over $[0,1]$: $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$. Denote $h_j = x_j - x_{j-1}$ $y_j = y(x_j)$, $f_j = f(x_j, y_j)$ etc. We write

$$(p(t))^{-1}(p(t)y')' = \frac{M_{j-1}}{h_j}(x_j - t) + \frac{M_j}{h_j}(t - x_{j-1}), \quad x_{j-1} < t < x_j. \tag{3}$$

It is obvious that

$$\left[\frac{1}{p(t)}(p(t)y')'\right]_{x_j} = M_j = f_j,$$

and

$$\left[\frac{1}{p(t)}(p(t)y')'\right]_{x_{j-1}} = M_{j-1} = f_{j-1}.$$

Multiplying Equation (3) by $p(x)$ and then integrating from τ to x_j, we get

$$p_j y_j' - p(\tau)y'(\tau) = \frac{M_{j-1}}{h_j}\int_{\tau}^{x_j}(x_j - t)p(t)dt + \frac{M_j}{h_j}\int_{\tau}^{x_j}(t - x_{j-1})p(t)dt. \qquad (4)$$

To get spline approximation in the interval $(x_{j-1},\ x_j)$, we divide equation (4) by $p(\tau)$, integrating from x_{j-1} to x and then setting interpolating condition $y(x_j) = y_j$, we get

$$y(x) = y_{j-1} - (S_j y_{j-1} - S_j y_j)\int_{x_{j-1}}^{x} p^{-1}(\tau)d\tau + \frac{M_j}{h_j}\left[S_j K_j^- \int_{x_{j-1}}^{x} p^{-1}(\tau)d\tau - \int_{x_{j-1}}^{x} p^{-1}(\tau)\int_{\tau}^{x_j}(t - x_{j-1})p(t)dt\, d\tau\right]$$

$$+\frac{M_{j-1}}{h_j}\left[S_j I_{1j}^- \int_{x_{j-1}}^{x} p^{-1}(\tau)d\tau - \int_{x_{j-1}}^{x} p^{-1}(\tau)\int_{\tau}^{x_j}(x_j - t)p(t)dt d\tau\right], \quad x_{j-1} < x < x_j \qquad (5)$$

where

$$S_j = \left[\int_{x_{j-1}}^{x_j} p^{-1}(\tau)d\tau\right]^{-1}, \qquad (6)$$

$$K_j^- = \int_{x_{j-1}}^{x_j}\left(\int_{x_{j-1}}^{t} p^{-1}(\tau)d\tau\right)(t - x_{j-1})p(t)dt,$$

$$I_{1j}^- = \int_{x_{j-1}}^{x_j}\left(\int_{x_{j-1}}^{t} p^{-1}(\tau)d\tau\right)(x_j - t)p(t)dt.$$

Setting $j = j + 1$ in equation (5) we get the spline in the interval $(x_j,\ x_{j+1})$. Now using Taylor's expansion

$$G(x) = G_j + (x - x_j)G_j' + \tfrac{1}{2}(x - x_j)^2 G_j'' + + \tfrac{1}{6}(x - x_j)^3 G'''(\zeta_j) \qquad (7)$$

where ζ_j lies between x and x_j, $G(x_j) = G_j$, $G'(x_j) = G_j'$, etc., we approximate K_j^-, I_{1j}^- and S_j as follows

$$K_j^- \approx (h_j A_{00,j}^- + A_{10,j}^-) + h_j(G_j'/G_j)(A_{10,j}^- + A_{01,j}^-),$$

$$I_{1j}^- \approx -A_{10,j}^-, \qquad (8)$$

$$\bar{S}_j = \left[G_j\left(\frac{x_j^{1-b} - x_{j-1}^{1-b}}{1-b}\right) + G_j'\left\{\frac{x_j^{2-b} - x_{j-1}^{2-b}}{2-b} - \frac{x_j(x_j^{1-b} - x_{j-1}^{1-b})}{1-b}\right\}\right]^{-1}.$$

Substituting expression given in (7), (8) in equation (5) and using continuity conditions for $y'(x)$ at mesh point x_j and setting $M_{j-1} = \bar{f}_{j-1}$, $M_j = \bar{f}_j$, $M_{j+1} = \bar{f}_{j+1}$, we get the three point finite difference approximation as:

$$-\bar{S}_j\tilde{y}_{j-1} + (\bar{S}_j + \bar{S}_{j+1})\tilde{y}_j - \bar{S}_{j+1}\tilde{y}_{j+1} = a_j\tilde{f}_{j+1} + b_j\tilde{f}_j + c_j\tilde{f}_{j-1}, \quad j = 1(1)(N-1) \qquad (9)$$

where

$$a_j = -A_{10,j}^+\,\bar{S}_{j+1}/h_{j+1},$$

$$b_j = -[(A_{00,j}^+\bar{S}_{j+1} + A_{00,j}^-\bar{S}_j) + (-A_{10,j}^+\,\bar{S}_{j+1}/h_{j+1} + A_{10,j}^-\,\bar{S}_j/h_j) + (G_j'/G_j)\{(-A_{10,j}^+ + A_{01,j}^+)\bar{S}_{j+1} + (A_{10,j}^- + A_{01,j}^-)\bar{S}_j\}],$$

$$c_j = -A_{10,j}^-\,\bar{S}_j/h_j,$$

with

$$A_{00,j}^{\pm} = \tfrac{1}{2(b+1)}x_{j\pm1}^2 - \tfrac{1}{2(1-b)}\left[\tfrac{2}{(b+1)}x_{j\pm1}^{1-b}x_j^{b+1} - x_j^2\right],$$

$$A_{01,j}^{\pm} = \tfrac{1}{6(b+2)}x_{j\pm1}^2(2x_{j\pm1} - 3x_j) - \tfrac{1}{(b+1)}x_{j\pm1}^{1-b}x_j^{b+1}\left[\tfrac{1}{(2-b)}x_{j\pm1} - \tfrac{1}{(1-b)}x_j\right] - \tfrac{(4-b)}{(1-b)(2-b)}x_j^3,$$

$$A_{10,j}^{\pm} = \tfrac{1}{3(b+2)}x_{j\pm1}^3 - \tfrac{1}{2(b+1)}x_{j\pm1}^2 x_j + \tfrac{1}{6(1-b)}\left[\tfrac{6}{(b+1)(b+2)}x_{j\pm1}^{1-b}x_j^{b+2} - x_j^3\right].$$

For the boundary condition $\alpha y(1) + \beta\, y'(1) = \gamma$, we require one more difference equation i.e. for $j = N$. Using equation (4) for the interval $(x_{N-1},\ x_N)$ and the boundary condition at $x = 1$, and setting $M_{N-1} = \bar{f}_{N-1}$, $M_N = \bar{f}_N$, we get

$$-\bar{S}_N\tilde{y}_{N-1} + (\bar{S}_N + \alpha/(\beta G_N))\tilde{y}_N = b_N\tilde{f}_N + c_N\tilde{f}_{N-1} + \gamma/\beta G_N, \qquad (10)$$

where

$$b_N = -\bar{S}_N[A_{00,N}^- + A_{10,N}^-/h_N + (G_N'/G_N)(-A_{10,N}^- + A_{01,N}^-)],$$

$c_N = -A^{-}_{10,j}\bar{S}_N/h_N.$

2.1 Truncation Error

Consider the mesh ratio parameter $\sigma_j = h_{j+1}/h_j$ then we have $x_{j-1} = x_j - h_j,\ x_{j+1} = x_j + \sigma_j h_j$. Using Taylor expansion given by equation (7) for $G(x)$ in equation (6) and substituting the expression for x_{j-1} and then expanding in Taylor series, we get,

$$S_j = \frac{x_j^b}{h_j G_j}\left[1 + \left(p_1 + \frac{1}{2}G_{1,j}\right)x' + \left(p_2 + \frac{1}{3}p_1 G_{1,j} + \frac{1}{4}G_{1,j}^2 - \frac{1}{6}G_{2,j}\right)(x')^2 + \cdots\right] \tag{11}$$

where $p_1 = -b/2$, $p_2 = -b(2-b)/12$, $x' = h_j/x_j$ and $G_{i,j} = x_j^i G_j^{(i)}/G_j,\ i = 1,2,3$. Similarly

$$S_{j+1} = \frac{x_j^b}{h_{j+1} G_j}\left[1 - \left(p_1 + \frac{1}{2}G_{1,j}\right)x^* + \left(p_2 + \frac{1}{3}p_1 G_{1,j} + \frac{1}{4}G_{1,j}^2 - \frac{1}{6}G_{2,j}\right)(x^*)^2 + \cdots\right] \tag{12}$$

where $x^* = \sigma_j h_j/x_j$. Now substituting the expressions for x_{j-1}, x_{j+1} in $A^{\pm}_{00,j}$, $A^{\pm}_{01,j}$, $A^{\pm}_{10,j}$ and expanding in Taylor series, we get the following:

$$A^{+}_{00,j} = h_{j+1}^2\left[\tfrac{1}{2!} - \tfrac{b}{3!}x^* + \tfrac{b(b+2)}{4!}(x^*)^2 - \cdots\right], \tag{13}$$

$$A^{+}_{01,j} = h_{j+1}^3\left[\tfrac{2}{3!} - \tfrac{3b}{4!}x^* + \tfrac{4b(b+2)}{5!}(x^*)^2 - \cdots\right], \tag{14}$$

$$A^{+}_{10,j} = h_{j+1}^3\left[\tfrac{1}{3!} - \tfrac{b}{4!}x^* + \tfrac{b(b+3)}{5!}(x^*)^2 - \cdots\right], \tag{15}$$

$$A^{-}_{00,j} = h_{j+1}^2\left[\tfrac{1}{2!} - \tfrac{b}{3!}x' + \tfrac{b(b+2)}{4!}(x')^2 - \cdots\right], \tag{16}$$

$$A^{-}_{01,j} = -h_{j+1}^3\left[\tfrac{2}{3!} - \tfrac{3b}{4!}x' + \tfrac{4b(b+2)}{5!}(x')^2 - \cdots\right], \tag{17}$$

$$A^{-}_{10,j} = -h_{j+1}^3\left[\tfrac{1}{3!} - \tfrac{b}{4!}x' + \tfrac{b(b+3)}{5!}(x')^2 - \cdots\right]. \tag{18}$$

Substituting (11)-(18) in equation (9) and simplifying then for difference scheme (9), we get the truncation error

$$t_j = \frac{1}{24 G_j}x_j^b(1+\sigma_j^3)h_j^3\left[f_j'' - \frac{G_j'}{G_j}f_j' + \left\{\left(\frac{G_j'}{G_j}\right)^2 - \frac{2G_j''}{G_j}\right\}f_j\right] + \cdots. \tag{19}$$

Similarly the truncation error t_N in (10) can be written as:

$$t_N = \frac{1}{24 G_N}h_N^3\left[f_N'' - \frac{G_N'}{G_N}f_N' + \left\{\left(\frac{G_N'}{G_N}\right)^2 - \frac{2G_N''}{G_N}\right\}f_N\right] + \cdots. \tag{20}$$

3 CONVERGENCE OF THE METHOD

In this section, we show that under quite general conditions this method is second order convergent. Let $F(\tilde{Y}) = (\tilde{f}_1,\ldots,\tilde{f}_N)^T$, $\tilde{Y} = (\tilde{y}_1,\ldots,\tilde{y}_N)^T$ and $R = (r_1,0,\ldots,0,r_N)^T$, then the difference scheme (9)-(10) can be expressed in matrix form as

$$S\tilde{Y} + MF(\tilde{Y}) = R, \tag{21}$$

where, $S = (s_{ij})$ and $M = (m_{ij})$ are $(N \times N)$ tridiagonal matrices with

$$s_{i,i-1} = -\bar{S}_i,\ i = 2(1)N,\ \ s_{i,i+1} = -\bar{S}_{i+1},\ i = 1(1)(N-1),\ s_{i,i} = (\bar{S}_i + \bar{S}_{i+1}),\ i = 1(1)(N-1),\ s_{N,N} = \bar{S}_N + \alpha/(\beta G_N),$$

$$m_{i,i} = -b_i,\ i = 1(1)N,\ m_{i,i-1} = -c_i,\ i = 2(1)N,\ m_{i,i+1} = -a_i,\ i = 1(1)(N-1),$$

$$r_1 = A\bar{S}_1 + c_1\tilde{f}_0,\ r_N = \gamma/\beta G_N.$$

Now let $Y = (y_1,\cdots,y_N)^T$ denote the exact solution then we can write

$$SY + MF(Y) + T = R. \tag{22}$$

From equation (21) and equation (22), we get the error equation

$$(S + MF)E = T, \tag{23}$$

where $T = (t_1,\cdots,t_N)^T$, $E = \tilde{Y} - Y = (e_1,\cdots,e_N)^T$, $F(\tilde{Y}) - F(Y) = FE$, $F = diag\{U_1,\cdots,U_N\}$, (where $U_i = \partial f_i/\partial y_i \ge 0$).

Since $S_j > 0,\ \ a_j < 0,\ b_j < 0,\ c_j < 0$ for sufficiently small mesh size, it is easy to see that $MF \ge 0$ and the matrices S and $S + MF$ are irreducible, monotone and hence S^{-1}, $(S+MF)^{-1}$ exists, is nonnegative and $(S+MF)^{-1} \le S^{-1}$.

Let $Z = (1, \cdots, 1)^T$and $W = (W_1, \cdots, W_N)^T = SZ$ denote the vector of row-sum of S. Also let $V = (V_1, \cdots, V_N)^T$, where $V_j = (2\beta/\alpha) + 2 - (x_j + 1)^2/2$ and $P = (P_1, \cdots, P_N)^T = SV$. We next obtain the bound for $S^{-1} = (s_{ij}^{-1})$. Since $P_1 > 0$, $P_N > 0$ and for sufficiently small h (uniform mesh size)

$$P_j > \frac{h}{G_j} x_j^b, \quad j = 2(1)(N-1),$$

then from $S^{-1}V = P$, we get,

$$\sum_{j=2}^{N-1} \frac{h}{2G_j} x_j^b s_{ij}^{-1} \le \max V_i < \frac{2\beta}{\alpha} + \frac{3}{2}, \quad i = 1(1)N. \tag{24}$$

Let there exist constants N_i, $i = 0(1)2$ such that $|f^{(i)}| \le N_i$, for $i = 0(1)2$, and $0 < x \le 1$ then for an uniform mesh and for sufficiently small h, we get,

$$|t_j| \le C \frac{h^3 x_j^b}{G_j}, \quad j = 1(1)(N-1), \tag{25}$$

and

$$|t_N| \le C \frac{h^3}{2G_N} \tag{26}$$

where

$$C = \frac{1}{6}\left[N_2 + N_1 \sup_{[0,1]}|G'(x)/G(x)| + N_0\{\sup_{[0,1]}|G'(x)/G(x)|^2 + 2\sup_{[0,1]}|G''(x)/G(x)|\}\right].$$

Now since $W_1 = \bar{S}_1$ and $W_N = \alpha/(\beta G_N)$, with the help of $S^{-1}W = z$, we obtain,

$$s_{i,1}^{-1} \le 1/W_1 = 1/\bar{S}_1 \implies s_{i,1}^{-1} \le x_1^{1-b} G_1/(1-b), \quad i = 1(1)N, \tag{27}$$

and

$$s_{i,N}^{-1} \le \beta G_N/\alpha, \quad i = 1(1)N. \tag{28}$$

In view of $(S + MF)^{-1} \le S^{-1}$, we have $\|E\|_\infty \le \|S^{-1}|T|\|_\infty$ and with the help of equations (24)-(28), we get,

$$|e_i| \le C^* h^2$$

where $C^* = C[1/(1-b) + 9\beta/2\alpha + 3]$ and hence

$$\|E\|_\infty = O(h^2).$$

Theorem 1 *Assume that $f(x,y)$ satisfies (**B**) and $p(x)$ satisfies conditions in (**A**). Then the difference scheme (9)-(10) for $b \in [0,1)$ based on an uniform mesh for the boundary value problem (1)-(2) are of second order accuracy for sufficiently small h provided f'', f' and f are bounded on $(0,1]$.*

4 NUMERICAL ILLUSTRATIONS

To illustrate the method and to verify the order of convergence of the method for general non-negative functions $p(x)$, we consider two example of singular boundary value problems.

Example 1

$$x^{-b}(x^b y')' = 5x^3(5x^5 - (b+4))y,$$

$$y(0) = 1,\ y(1) + y'(1) = 6e,$$

with exact solution $y(x) = \exp(x^5)$.

Example 2

$$(x^b e^x)^{-1}(x^b e^x y')' = 5x^3(5x^5 e^y - (b+4) - x)/(4 + x^5),$$

$$y(0) = -\ln 4\,1,\ y(1) + 5y'(1) = -\ln 5 - 5,$$

with exact solution $y(x) = -\ln(4 + x^5)$.

Table I: Maximum absolute error for example-1

N/b	0.25	Order	0.50	Order	0.75	Order
16	4.99(-2)[a]	-	5.06(-2)	-	5.13(-2)	-
32	1.29(-2)	1.95	1.31(-2)	1.95	1.33(-2)	1.95
64	3.24(-3)	1.99	3.29(-3)	1.99	3.34(-3)	1.99
128	8.12(-4)	2.00	2.85(-4)	2.00	8.35(-4)	2.00
256	2.03(-4)	2.00	2.06(-4)	2.00	2.09(-4)	2.00
512	5.08(-5)	2.00	5.16(-5)	2.00	5.22(-5)	2.00

[a]4.99(-2) = 4.99 × 10^{-2}

Table II: Maximum absolute error for example-2

N/b	0.25	Order	0.50	Order	0.75	Order
16	4.73(-3)	-	6.25(-3)	-	8.48(-3)	-
32	1.19(-3)	1.99	1.57(-3)	1.99	2.13(-3)	1.99
64	2.97(-4)	2.00	3.93(-4)	2.00	5.33(-4)	2.00
128	7.43(-5)	2.00	9.82(-5)	2.00	1.33(-4)	2.00
256	1.86(-5)	2.00	2.46(-5)	2.00	3.33(-5)	2.00
512	4.65(-6)	2.00	6.14(-6)	2.00	8.33(-6)	2.00

Maximum absolute errors and order of convergence (accuary) for examples 1 and 2, have been displayed in Tables I and II respectively for three value $0.25, 0.50, 0.75$ of b, which show that the method works well and is of second-order accuracy.

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