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A Linking Visual Active Representation DHLP for Student's Cognitive Development

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A LINKING VISUAL ACTIVE REPRESENTATION DHLP FOR STUDENTS COGNITIVE DEVELOPMENT

Strictly as per the compliance and regulations of:

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Keywords : Hypothetical learning path, dynamic geometry software, linking visual active representations (LVAR).

I. Introduction

In the sections that follow, I shall describe a 'dynamic' hypothetical learning path (DHLP) (i.e, a hypothetical learning path through the dynamic geometry software) for the learning of the concept of parallelogram in geometry, which I "designed to engender those mental processes or actions [of students] hypothesized to move [them] through a developmental progression of levels of thinking" (Clements & Sarama, 2004, p.83). Simon (1995) supports that a hypothetical learning trajectory "is hypothetical because the actual learning trajectory is not knowable in advance" (p. 135).

As a mathematics teacher, I have designed instructional materials for my students in the past (see for example Patsiomitou, 2005, 2007), endeavouring to predict students thinking, or "imagining a route by which [the student] could have arrived (or could arrive) at a personal solution" (Gravemeijer & Terwel, 2000, p.780). This is in accordance to the "reinvention principle" (Freudenthal, 1973) or working in a DGS environment in accordance to the 'dynamic reinvention' principle (Patsiomitou & Emvalotis, 2010a, b; Patsiomitou, Barkatsas & Emvalotis, 2010) Furthermore, "an individual's learning has some similarity to [the learning] that many of the students in the same class can benefit from the same mathematical task" (Simon, 1995, p. 135).

Students' cognitive growth is a major aim of mathematics education. Researchers have interpreted it in different ways, such as that cognitive growth can occur between others, through developmental stages (e.g., Piaget, 1937/1971; van Hiele, 1986), as development of proof schemes (e.g., Balacheff, 1988; Harel & Sowder, 1998; Harel, 2008) or as dynamical development of students' mental representations (e.g., Cifarelli, 1998) when students confront problem-solving situations.

Pegg & Tall (2005) identify two main categories of theories to explain and predict students' conceptual development, (or cognitive growth, or cognitive development):

- *"global theories of long-term growth* of the individual, such as the stage theory of Piaget (e.g., Piaget & Garcia, 1983).
- *local theories of conceptual growth* such as the action-process-object-schema theory of Dubinsky (Czarnocha et al., 1999) or the unistructural multistructural-relational-extended abstract sequence of SOLO Model (Structure of Observed Learning Outcomes, Biggs & Collis, 1982, 1991; Pegg, 2003)". (p.188)

In the present study I have used the theory of van Hiele (1986) (a long-term or global theory in terms of Pegg & Tall's (2005) identification and categorization mentioned above) both in the design of the activities in the DGS environment in the light of "the path by which learning might proceed" (Simon, 1995, p.135) and for describing of student's behaviour.

The students during a problem-solving situation and due to the communication that develops in a mediated-by-artifacts milieu, the students are led to create their personal representations for a mathematical entity and to transform them. In order to develop the understanding of a meaning the students have to create a transitional bridge between the external and internal representation (e.g, Kaput, 1999; Goldin & Shteingold, 2001; Pape & Tchoshanov, 2001) of this meaning. The activity of solving problems is based on the interaction and transformation between different representational systems (e.g, Goldin & Janvier, 1998) of the same

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meaning. The ability to interpret a meaning between representational systems (Janvier, 1987) is necessary for students' conceptual understanding in mathematics.

In previous studies I have supported the effect that the Linking Visual Active Representation modes (see for example Patsiomitou, 2008 a, b; Patsiomitou, 2010) have on the student's gradual competence towards rigorous proof construction, during a problem solving process. In Geometer's Sketchpad (Jackiw, 1991) DGS environment, LVAR are interpreted as a real-world problem modeling process "encoding the properties and relationships for a represented world consisting of mathematical structures or concepts" (Sedig & Sumner, 2006) enhanced by selected basic or task – based (Sedig & Sumner, 2006) different interaction techniques facilitated by the DG Sketchpad v4 environment where the problem is modeled (see also Patsiomitou, 2008b, 2010, Patsiomitou & Emvalotis, 2009b). "Linking Visual Active Representations" [1] during a dynamic geometry problem solving session are defined as follows (e.g, Patsiomitou, 2008, 2010):

Linking Visual Active Representations are the successive phases of the dynamic representations of a problem which link together the problem's constructional, transformed representational steps in order to reveal an ever increasing constructive complexity. Since the representations build on what has come before, each one is more complex, and more integrated than the previous ones, due to the student's (or teacher's, in a semi-preconstructed activity) choice of interaction techniques during the problem-solving process, aiming to externalize the transformational steps they have visualized mentally (or existing in their mind) (p. 2).

In this study, I shall extend the conceptual frame of the Linking Visual Active Representations in order to include what emanated from the research process through out in depth data analysis. What I shall prove through the current study is (a) how crucial is the development of student's ability to decode a representation either mental or external and (b) how the linking representations which a student mentally creates, affect his/her development of geometrical thinking. Thus, a student's thinking development could be evoking in an organized frame of a learning path in which the student participates.

As it is well known the development of student's thinking is depended on the structure of the content of the teaching process. From that point of view the structure of the design of the activities and their sequence during the implementation process plays the main role. The student's learning using LVAR through their participation in a hypothetical leaning path can change the path of student's development due to their reconceptualization of the meanings that will be introduced.

In the next sections I shall describe in detail both how the students might interact with the instructional materials of the DHLP and what their hypothetical learning path, goals and predictable modes of thought might be. I shall also present snapshots of the research process. The goal of my study was to investigate the research question:

Does the DHLP ('dynamic' hypothetical learning path) supported by LVARs (Linking Visual Active Representations) affect students' cognitive development?

II. Theoretical Framework

a) The Van Hiele Model

Dina and Pierre van Hiele-Geldof developed a theoretical model involving five levels of thought in geometry and five phases of instructional design after they observed the great difficulties that secondary school students experienced when learning geometry (in Fuys et al., 1984, p.6). Pierre van Hiele eventually reduced their model to three levels: visual (level 1), descriptive (level 2) and theoretical (level 3) (see van Hiele, 1986 cited in Teppo, 1991, p. 210). Battista (2007) "has elaborated the original van Hiele levels to carefully trace students' progress in moving from informal intuitive conceptualizations of 2D geometric shapes to the formal property-based conceptual system used by mathematicians" (p.851).

He separated each phase in subphases (Battista, 2007):

Level 1 (Visual-Holistic Reasoning) is separated into sublevel 1.1. (prerecognition) and sublevel 1.2 (recognition).

Level 2 (Analytic-Componential Reasoning) is separated into sublevel 2.1 (Visual-informal componential reasoning), sublevel 2.2 (Informal and insufficient-formal componential reasoning) sublevel 2.3 (Sufficient formal property-based reasoning)

Level 3 (Relational –Inferential Property-Based Reasoning) into sublevel 3.1 (Empirical relations), sublevel 3.2 (Componential analysis), sublevel 3.3 (Logical inference) and sublevel 3.4 (Hierarchical shape, classification based on logical inference).

Level 4 (Formal Deductive Proof) (pp.851-852)

Another aspect of van Hiele's theory is the importance of students adhering to the following five instructional phases within each level which are briefly the following (Fuys et al., 1984): information (inquiry), directed orientation, explicitation, free orientation and integration (p.251).

Teppo (1991) supports that "students progress from one level to the next is the result of purposeful instruction […] that emphasize exploration, discussion, and integration" (p. 212).

1 Examples of Linking Visual Active Representations are given later in the text .

As Pierre van Hiele reports "an important part of the roots of his work can be found in the theories of Piaget" (van Hiele, 1986, p. 5). Pierre van Hiele also reported the differences between his theory and the theory of Piaget, giving emphasis to the role of language "in moving from one level to the next" (van Hiele, 1986, p. 5). He also saw "structures of a higher level [thought] as the result of study of the lower level" (van Hiele, 1986, p. 6).

During the instructional phases the figures firstly acquire the symbol character and after a successful instructional period in which the student participates the figures acquire the signal character (Pierre van Hiele, 1986; Sang Sook Choi-Koh, 1999; Cannizzaro & Menghini, 2003; Patsiomitou & Emvalotis, 2010 a, b). Meaning the student transforms "a first level of perception at which pupils condense the properties of a known geometrical figure" to "a second level of description or analysis at which perceptions are translated into descriptions, though without specific linguistic properties—of which the significant signal is most significant in the description" (Cannizzaro & Menghini, 2003, p.2).

The students in the gaps between levels face disequilibration (Piaget, 1937/1954) situations that force them to reorganize their cognitive structures, when a conceptual structure does not act in line with their expectations. The reorganization of the individual's schemata involves the subprocesses of accommodation or assimilation (Piaget, ibid.) which correspond to modifying the pre-existing schemata and building new schemata in the student's mind or interpreting the new information according to pre-existing schemata. Many times students face misconceptions (e.g, Shaughnessy, 1981) and cognitive conflicts (e.g., Watson & Moritz, 2001).

The difficulties which arise when a student studies geometry begin with the way s/he perceives a shape. The perceptual competence of a student to 'see' a figure's properties depends on his/her development of cognitive structures and ability to think abstractly. The development of a student's cognitive structures makes him/her able to perform the "hypothetical representation of his/her internalized organization of the concepts in long-term memory" (McDonald, 1989, p.426).

Skemp's view of the abstraction process is that "a concept is the end product of […] an activity by which we become aware of similarities […] among our experiences" (Skemp, 1986, p.21 in White & Mitchelmore, 2010, p.206). Moreover, Schwartz, Herschkowitz & Dreyfus (2001) argue that *"[…] Abstraction is not an objective, universal process but depends strongly on context, on the history of the participants in the activity of abstraction and on artifacts available to the participants. Artifacts are outcomes of human activity that can be used in further activities. They include material objects and tools, such as computerized ones, as well as mental ones including language and procedures; in particular, they can be ideas or other outcomes of previous actions" (p.82).*

Dina van Hiele made clear in her writings the distinction between the 'drawing' and the 'construction' of a shape. She distinguished the notion of construction from the notion of drawing in order to express the difference between the images that a student constructs (in a paper/pencil environment) when s/he tries to externalize his/her mental representation, using geometry rules (or not in correspondence). She supported that "the teacher [in order] to reach his goal [has] to refine [to his/her students] that there is a clear distinction between the drawing of figures and the constructing of figures" (Fuys et al., 1984, p. 36). In other words it is crucial for the students' cognitive development to improve their ability to transform the visual image or drawing they perceive, into a construction with concrete properties. The investigation of problems in the dynamic geometry environment provides the feedback for the students to acquire a theoretical background, necessary for the conceptual development in Euclidean geometry. During the problem-solving process, students develop different kinds of reasoning including inductive, abductive, plausible and transformational reasoning (e.g, Harel & Sowder, 1998; Peirce, 1992; Simon, 1996).

As for procedural knowledge Baroody, Feil & Johnson (2007) define it as the "mental actions or manipulations, including rules, strategies, and algorithms, needed to complete a task." (p. 123). Kadijevich & Haapasalo (2001) argue that, using computers, students can spend less time on procedural skills and more on developing their conceptual understanding (Fey, 1989). Given the core role in mathematics education of developing procedural and conceptual knowledge and forging links between the two, a key question is how different technologies affect the relationship between the two.

Laborde (2005) has distinguished between robust and soft constructions, placing emphasis on difficulties of students to connect their construction with the theory of geometry, in other words to relate the procedural knowledge and conceptual understanding.

In a DGS milieu "robust constructions are constructions for which the drag mode preserves their properties" (Laborde, 2005, p.22).

The solution of a problem in a DGS environment depends on the preexisting conceptual knowledge of students about figure and their procedural knowledge of the tools and theorems which might be used, moreover the tools' efficiencies. Furthermore, conceptual knowledge of students emanates in response to instrumental genesis (e.g., Rabardel, 1995) through the tool use of the software and the development of argumentation as a discursive process, supported by the visualization provided by the dynamic diagram.

During the instrumental genesis the user structures what Rabardel (1995) calls utilization schemes (usage schemes or instrumented action schemes) of the tool/artifact. Utilization schemes are the mental schemes that organize the activity through the tool/artifact. This process has been reported by many studies (e.g, Artigue, 2000; Trouche, 2004) on the research of Verillon & Rabardel (1995) about the ways by which an artefact becomes an instrument for a student. According to Artigue (2000), "an instrument is thus seen as a mixed entity, constituted on the one hand of an artefact and, on the other hand, of the schemes that make it an instrument for a specific person. These schemes result from personal constructions but also from the appropriation of socially pre-existing schemes." Vergnaud (1998) has redefined the meaning of scheme that has been introduced by Piaget (1936), as the "invariant organization of behaviour for a given class of situations". From Trouche's point of view, (personal e-mail correspondence with Professor Trouche on October 22, 2007) "[someone] has also to have in mind social aspects of schemes. And, finally, what is important is to analyze the operational invariants, behind the schemes…". Meaning "the implicit knowledge contained in the schemes: concepts-in-actions, that is concepts that are implicitly considered as pertinent, or theorems-in-actions that is, propositions believed to be true" (Trouche, 2004, p. 285).

Dragging is a powerful, conceptual tool in a DG milieu which that does not have "compatible counterpart" in Euclidean geometry (Lopez-Real & Leung, 2004, p.1). According to Mariotti (2000, p.36) "the dragging test, externally oriented at first, is aimed at testing perceptually the correctness of the drawing; as soon as it becomes part of interpersonal activities […] it changes its function and becomes a sign referring to a meaning, the meaning of the theoretical correctness of the figure."

In a current study (Patsiomitou, 2011) I introduced the notions of theoretical dragging (i.e., the student aims to transform a drawing into a figure on screen, meaning s/he intentionally transforms a drawing to acquire additional properties) and experimental dragging (i.e., the student investigates whether the figure (or drawing) has certain properties or whether the modification of the drawing in the picture plane through dragging leads to the construction of another figure or drawing). I also reported of the notion of instrumental decoding to explain a student's competence to transform his/her mental images to actions in the software, using the software's interaction techniques.

In this study I shall describe how the learning through the DHLP affects students' cognitive structure's transformations and consequently their cognitive growth. I shall also explain how the theoretical dragging affects students' competence to instrumental decoding and consequently their cognitive development.

b) Learning As a 'Dynamic' Reinvention

The theoretical framework underpinning the DHLP was based on social constructivism. In a social - constructivist teaching and learning process, the learning of mathematics generally and of geometry particularly is a complex process, being constructivist and social (Cobb, Yackel & Wood 1989; Yackel, Cobb, Wood, Wheatley & Merkel 1990; Cobb & Bauersfeld 1995; Yackel, Rasmussen & King 2001; Yackel & Rasmussen 2002; Jaworski, 2003).

Many researchers (for example Goos, Galbraith & Renshaw, 2002; Dekker & Elshout-Mohr, 2004) recognise the "potential of working in small groups" (Dekker & Elshout-Mohr, 2004, p. 39). Moreover, the mathematical discourses developed in a small group mediated by cognitive tools such as the Geometer's Sketchpad regulate the social interactions and enhance students' mathematical communication. According to Sfard (2001)

"Most of our learning is nothing else than a special kind of social interaction aimed at modification of other social interactions. […] Thus, whatever the topic of learning, the teacher's task is to modify and exchange the existing discourse rather than to create a new one form scratch. If so, we can define learning as the *process of changing one's discursive ways in a certain well-defined manner."* (p.3)

In other words, this will be a change in a student's informal discursive way to express his or her thoughts in formal language. Building on a theoretical perspective of learning, Bowers & Stephens (2011) support that *first, if learning is viewed as a socially situated practice, then (a) teaching can be seen as the practice of orchestrating mathematical discourses and (b) learning can be seen as the ways in which students engage in these discourses. In short, the role of any teacher (or teacher educator) can be seen as negotiating the emergence of conceptual discourse that involves the use of appropriate tools as a normative part of the commognitive process. The role of the student is also intricately related to his or her participation in the discourse with a focus on the ways in which tools mediate the discussions and acceptable ways of proffering and debating mathematical ideas. (p. 287)*

In such a discursive process the students play the role of the 'actor' in the activity of the mathematical discussion and the teacher the role of the participated 'observer", who frequently intervenes with crucial questions designed to prompt mathematical discussion. Freudenthal (1991) "criticized the constructivist epistemology from an observer's point of view" [and] "saw mathematics from an actor's point of view" (Gravemeijer & Terwel, 2000, p.785). Which is to say, constructing meaningful activities for the students by imagining how the students might interact with the instructional materials, what obstacles they had to overcome, the possible (or multiple) solutions they could find, how their thinking could be raised due to the evolution of mathematical discussions they participate in. This is in accordance with what Freudenthal argues that "doing mathematics is more important than mathematics as a ready-made product" (Gravemeijer & Terwel, 2000, p.780). In accordance to Steffe & Olive (1996), Olive (1999), Olive & Steffe (2002), Olive & Makar (2010) the mathematical knowledge which children build up during their engagement in a mathematical activity, is distinguished among others to

- 'children's mathematics – the mathematics that children [...] construct for themselves and is available to them as they engage in mathematical activity';
- 'mathematics for children – the mathematical activities that curriculum developers/writers and teachers design to engage students in meaningful mathematical activity' (Olive & Makar, 2010, p.136)

Freudenthal (1991) spoke of 'guided reinvention' to mention the kind of knowledge the students could acquire "as their own, personal knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Terwel, 2000, p.786). On the other hand "the teachers should be given the opportunity [to their students to] build their own mathematical knowledge-store on the basis of such a learning process" (Gravemeijer & Terwel, 2000, p.786). Many researchers argue that working in a dynamic geometry environment allows students to reinvent their personal knowledge by interacting with the other members of the group or with the teacher (or the participating researcher). For example, Furringhetti & Paola (2003) support that "in this case, the reinvention is guided, [...] by the use of the [dynamic geometry] environment". In the current study it will be investigated where the DGS environment affected students' dynamic reinvention of knowledge (Patsiomitou& Emvalotis, 2010a, b; Patsiomitou, Barkatsas & Emvalotis, 2011).

Building on the ideas mentioned above I think that dynamic reinvention of knowledge is the kind of knowledge the students could reinvent by interacting with the artefacts made in a DGS environment, "knowledge for which they themselves are responsible" (Gravemeijer & Terwel, ibid.)

c) The Pseudo-Toulmin's Model

Toulmin's (1958) model of argumentation is a model which relates the involved elements: claims, data, warrants, backings, qualifiers and rebuttals in the argument formulated by an individual (or a group of students that participate). These elements are represented in a diagram below in which the relationships between them are expressed in sequential order.

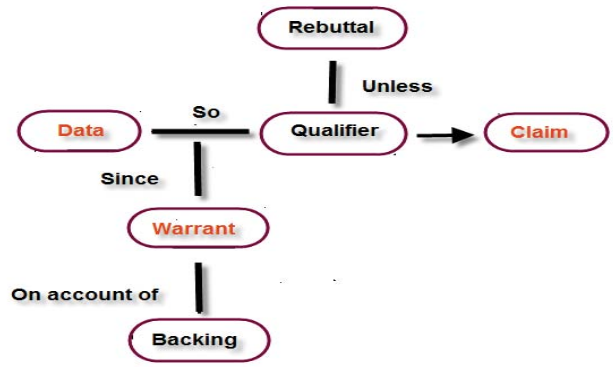


Fig. 1. Toulmin's (1958) model of argumentation (adapted)

In other words Toulmin's model consists of the elements described above, which are explicit or implicit. Several times an argument does not include qualifiers and rebuttals. Krummheuer (1995) suggested and applied a reduced model of the original scheme, consisting of claims, data, and warrants of arguments "to examine the learning of mathematics in the context of collective argumentation" (p.11). As suggested by Krummheuer (ibid.), during a classroom activity (or for the current study during group cooperation) one or more students could be contributing towards the formulation of the argument, attempting to convince the other participants of the group, including the class teacher (or the researcher). In the following paragraphs, I am going to explain the pseudo- Toulmin's model through examples in which

- the data could be an element or an object of the dynamic diagram, and
- a warrant could be a tool or a command that guarantees the result which is the claim (or the resulted formulation).

The figure 2 presents a pseudo-Toulmin's model through example.

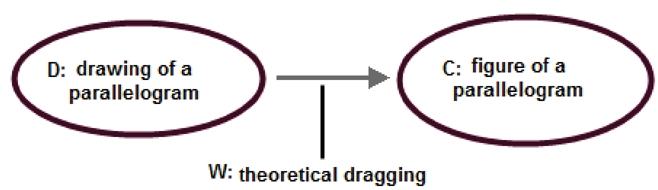


Fig.2. An example of a reduced pseudo-Toulmin's model

In the Figure 2, a drawing of a parallelogram is the data (D), the theoretical dragging is the warrant (W), and the figure of the parallelogram is the claim (C). This means that a student can theoretically drag a point-vertex of a drawing-parallelogram and transform it into a figureparallelogram, trying to acquire additional properties.

Also, I have extended the pseudo-Toulmin's model in order to express a relationship between the figures, something that I am going to present in the next sections.

III. Research Design

The qualitative study (Merriam, 1998) with a quasi-experimental design (Campbell & Stanley, 1963) was conducted in a public high school class in Athens during the second term of the 2006– 2007 academic years. For the research process twenty eight students volunteers were divided into 'experimental' and 'control' teams, of 14 students each. Students were ages 15 and 16, equal numbers of boys and girls, and all in levels 1 and 2. The students first had been evaluated by their responses to the 20 questions of the 25 multiple-choice questions van Hiele test of Usiskin (1982). In grading the students tests, "a student was assigned [the] weighted high score" described by Usiskin (1982, pp.22-23). This means s/he had been determined to be in level 1 if s/he answered 3 or 4 of 5 first questions of the Usiskin test correctly (with 4 being the stricter criterion called for by Usiskin). The participants had no knowledge of the DGS software or any related software.

The study developed into a didactic experiment of action research (Kemmis & McTaggart, 1982; Schön, 1987). For this study, the constant comparative method was chosen in order to deduce a grounded theory (Strauss & Corbin, 1990).

The students of the experimental group followed a DHLP (i.e. a re-conceptualized learning path of four strands for the teaching and learning of parallelograms in geometry, using The Geometer's Sketchpad software) which I conceived through a thought experiment, reported at many conferences [see for example Patsiomitou & Emvalotis, 2009c, 2010]. The students in the control group followed the class curriculum. The progress of both groups was evaluated with scheduled tests at intervals and at the end of the academic year. The aim of the study was to investigate if the students who had followed the DHLP could develop their thinking and to compare their development with the development of the control group, which had not followed the DHLP. The complete study includes the following investigations:

a) A detailed investigation of four phases of the students of the experimental group that followed the DHLP. Investigation covered how every student of the experimental group developed his/her thinking, using a detailed analysis of their formulations and comparing the kind of representations they produced and the kinds of definitions and reasoning (i.e., inductive, abductive or deductive).
b) A detailed investigation of four evaluations of the students of both groups in a paper-pencil environment. This investigation covered how every student in both groups developed his/her thinking by comparing the milestones of their development moving through the van Hiele levels (i.e., the characteristics of every level as defined by Battista (2007) as they appeared in the paper-pencil tests). Moreover, I studied their ability to prove.
c) A comparison study between the students in both groups (i.e., how the students in level 1 or level 2 of the experimental group developed the characteristics of each level and how members of the control group did the same).

In the current study, I shall concretely report the design and redesign of the DHLP (in more detail for phases A and B) through linking visually active representations and the experimental group students' competence in mental or verbal decoding of these representations and in using the tools that affect their development of thinking levels. The study of the control group is not the aim of the current paper, but I shall briefly discuss its development.

The phases of the DHLP are interconnected in terms of: a) the conceptual context, b) the order in which the software's technological tools are introduced, and c) the increasing difficulty at both levels. The experimental process lasted approximately four months, from January to May. Firstly I examined student's level of geometric thought using the test developed by Usiskin (1982) which is in accordance to the van Hiele model using only the first twenty questions of the questionnaire. The results presented here emerged from interaction within the group of the experimental team, with reference to excerpts from all four research phases. In the next sections, I shall describe the DHLP. This description of the DHLP is a synthesis of an instructional design process and a redesign process, meaning a "systematic, self reflective spiral of planning, acting, observing and reflecting" (Steketee, 2004, p. 876).

- In the instructional design process, I shall describe how I predicted the hypothetical transitional understanding of the meaning of parallelograms and the students' way of thinking during the solution of the problems in combination with their actions in the software with the closest possible approach.
- In the instructional redesign process, I shall describe the procedures that demanded the addition of new tools, which helped the students of the experimental team overcome cognitive and instrumental obstacles that they faced during the research process.

The description that follows is separated into two sections for each phase:

- one which describes the aims of the DHLP as part of the general framework of the curriculum for the teaching and learning of geometry, and
- a prediction process of the hypothetical interactions of the students with the tools, consequently an

inductive way of thinking that has been supported by my previous observations.

In the next sections I present excerpts of the research process concerning the groups A- E. In group A, the student participants were M9, M10, and M14 (all van Hiele level 1 at the pre-test). In group B were M1 and M12 (both van Hiele level 1 at the pre-test) and M11 and M12 (both in van Hiele level 2 at the pre-test). In group C student participants were M7, M8 (both in van Hiele level 2 at the pre-test) and M13 (van Hiele level 1 at the pre-test). In group D student participants were M5, M6 (both in van Hiele level 2 at the pre-test). In group E were M3 and M4 (both in van Hiele level 2 at the pre-test). During description of the DHLP will present snapshots of how the student–participants reacted with the digital artefacts and how their reactions gave me feedback to redesign the research process.

IV. The Description of the Hypothetical Learning path

a) *Phase A: Building and transforming figures through Linking Visual Active Representations*

i. *Instructional Design Process :* The aim of the first phase of the research process was for the students to obtain the competence to build and transform linking structurally unmodified representations of parallelograms. The groups started with the most general concept of a parallelogram in which the opposite sides are parallel lines, before specifying by imposing the properties that produce a rectangle, a rhombus, and a square.

In the first phase of the research process the students had to build parallelograms with an emphasis on the "construction" menu. My intention was to introduce the Sketchpad tools and commands 'step by step', "in parallel with the corresponding theory" (Mariotti, 2000, p.41), because from my previous experience the students too often make mechanical use of the software and, this in return renders them unable to understand the logic behind the command options. I have recorded in detail how the students came to understand the use of the tools and correlated this ability with the partial construction of the meanings. The aim of the construction problems of the research process was for the students to do the following:

- Construct a soft construction and investigate it using experimental dragging in order to face cognitive conflicts.
- Become able to dynamically reinvent the properties of the shapes through theoretical dragging.
- Provide a robust construction by instrumentally decoding their mental images with the software tools. The students have to first transform the verbal or written formulation ("construct a parallelogram," for example) into a mental image, which is to say an internal representation recalling a prototype image (e.g., Hershkovitz, 1990) that they have shaped from a textbook or other authority before transforming it into an external representation, namely an on-screen construction.
- Provide an oral description of the process, meaning the path they followed in constructing the figure. This process includes the relation of procedural knowledge (use of the tools, use of the theorems or definitions) with the students' conceptual understanding, meaning the use and building of the relative meanings through the process.
- Become able to perceptually form a hierarchy of the figures through linking representations.

The connection with the conceptual knowledge will occur as a result of the justification of the process "providing good arguments which can make the solution acceptable" (Mariotti, 2000, p. 34) at the theoretical field of the software within the system of Euclidean Geometry. As a consequence, "solving construction problems in the [DGS] environment means accepting not only all the graphic facilities of the software, but also accepting a logic system in which its observable phenomena will make sense" (Mariotti, 2000, p.28).

1. *Problem 1 : Construction of a parallelogram: construct a parallelogram if you know a straight line segment and a point on the screen*[2]*.*

Design process : a) When reading the problem, most students will start constructing a drawing on screen as an interpretation of the mental representation they have constructed by interacting with geometry curriculum materials (for example, textbooks). Due to experimental dragging which the student applies on a vertex, this drawing is messed up (Fig. 3). Through this process and in response to instrumental genesis the student will face a cognitive conflict between what s/he knows of the concept of a parallelogram, meaning what s/he has constructed from an authority (for example, a textbook) and what s/he faces on screen.

The transformation of the position of the point-vertex through theoretical dragging leads to the transformation of the segment in order for the opposite sides to become congruent (Fig. 6). The tool thus affects the students' understanding that opposite sides of a concrete parallelogram should be congruent. This is to say, the students dynamically reinvent their understanding through the process.

2 The students were allowed to lengthen the line, but not allowed to draw a parallelogram on the screen without using the given line and the concrete point.

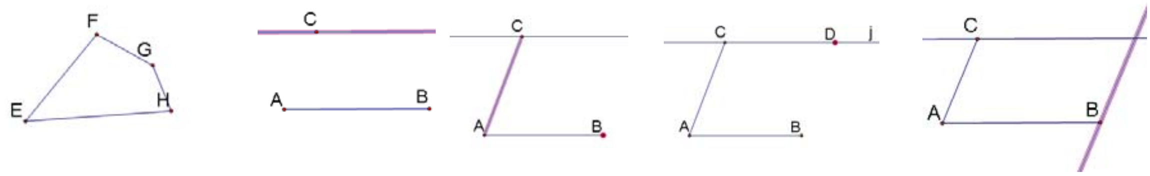


Fig.3 *Fig.4* *Fig.5* *Fig.6* *Fig.7*

Building Linking Visual Active Representations of a parallelogram

An example of the research process includes the following discussion:

M10: How can it become a parallelogram?
M14: drags a point-vertex of the drawing on screen.
M10 : Yes, but we don't know that this is a parallelogram like the one she made.
M14: It seems to be a parallelogram.
M9 : Maybe the 'dot' should be closer. (fig. 6)
M10 : It is not a parallelogram.
M9 : Oh, [this is not a parallelogram because] these parallel lines are not congruent!

Through the theoretical dragging of the point tool, M9 added the drawing of the parallelogram to the property of the congruency of its opposite sides. Subsequently, the theoretical dragging mediated the dynamic reinvention of a property of the diagram. In other words, theoretical dragging mediated the forming of an iconic representation and then the interpretation of the iconic representation into a verbal one.

Consequently, the dragging tool will modify the shape (for example a drawing-parallelogram is modified into a quadrilateral and then into a figure-parallelogram); challenged to reproduce the external representation, the students will seek a procedure to produce a robust construction. This can be achieved through the process of instrumental decoding mentioned above and by constructing usage schemes using the software's tools.

b) Through the process of instrumental decoding and seeking a procedure leading to an unmodified construction, students will use the software's primitives and commands to construct parallel lines. The notion of parallelism of lines "is necessary in order to obtain a geometrical structuring" (Fuys et al., 1984, p.161) that could not be acquired by the students at the first stages of the experimental process. According to Laborde (2003) "in a compass and ruler construction in paper and pencil environment, students would use a strategy based on the congruence of opposite sides. But in Cabri, almost all students use the strategy of constructing parallel lines to the given segments in order to obtain the fourth vertex C" (p.2)

Most processes require the student to think concretely with regard to how they conceive of an isolated line or an isolated point, or a line or lines belonging to a figure. According to Mesquita, (1998) "an isolated line and the same line belonging to a figure are not the 'same' to the perception. The identification of these two functions to the same line presupposes an analytic perception, which is not natural" (Merleau-Ponty, 1945, p. 18 in Mesquita, 1998, p. 184).

2. *Problem 2 : Construction of a rectangle. Drag the vertex of the parallelogram you have constructed until it becomes a rectangle. Then, find a way to construct a robust construction of a rectangle.*[3]

Design process : a) The rectangle is a fundamental meaning in parallelograms. Students are able to recognize the prototype image of the rectangle from the first classes of primary school. The obstacles regarding the prototype image of the rectangle have broadly been discussed (Hasegawa, 1997; De Villiers, 1994; Laborde, 1994; Fischbein, 1993; Parzysz, 1991; Sfard, 1991; Hershkovitz, 1990 in Monaghan, 2000, p. 187).

Most students "recognize a rectangle where the vertical width is greater than the horizontal length [...]. This is not how students perceive it, however; their concept of a rectangle has become fixed as being synonymous with an oblong[4]. [...].This perception, of course, is commonly held but is mathematically inaccurate as it ignores the square as a special case of rectangle" (Monaghan, 2000, pp. 186-187).

Through the experimental, and then theoretical, dragging of a vertex of parallelogram is pursued /seeked the students to focus on the figure's structure that "can be specialized [from a parallelogram] by imposing more properties" (De Villiers, 1994, p. 14) and can be generalized from the concept of square. The students will specialize on a structure of a parallelogram as "component structure of a higher one, [...], and they will learn to recognize corresponding elements, by acquiring the structure of a technical language" (Dina van Hiele, 1984, p. 187).

By this process, the students will construct the meaning of the rectangle as a specialization of the meaning of the parallelogram, incorporating the additional properties of the rectangle, which will be reinvented through the process; this means 'dynamically reinvented' (Fig. 8).

An example of the research process includes the following discussion:

M11: drags the vertexes of the parallelogram.
M11: Now it seems like a rectangle.
M11: I can't find exactly the point.

The experimental dragging of the vertexes of the parallelogram helped M11 to form a mental construct of the rectangle as a parallelogram. In response to instrumental genesis, she dynamically reinvented the property of the congruent sides. The synthesis of the interaction of the dragging tool on the point tool mediated into the transformation of her mental and verbal representations as an iconic representation.

3. *Problem 3: Construction of a rhombus. Join the opposite vertices of the parallelogram you constructed earlier[5]. Drag one vertex until you construct a rhombus. What did you observe? Then, construct a robust rhombus.*

Design process : a) The students will shape the drawing of a rhombus by theoretically dragging the parallelogram so that the figure will obtain the property of the congruency of the sides and will match the mental prototype image the students have for the figure of the rhombus. Dragging the vertices of the parallelogram, linking representations are shaped, which help students perceptually understand the rhombus as a specialized parallelogram. The theoretical dragging of a rhombus vertex will encourage them to consider perceptual hierarchy, i.e., the rhombus is a specialized concept of the parallelogram and a generalized concept of the square. Moreover, the students simultaneously visualize the rhombus as a synthesis of two isosceles triangles, something that I expected because I had observed it in the past from many other students who constructed the rhombus (Fig. 8).

b) A second intended activity will be for the students to theoretically drag the figure of the rhombus so that the isosceles triangles become equilaterals[6]. The The perception of the rhombus as a synthesis of two equilateral triangles may lead students to a cognitive conflict. For example, a student of level 1 is not able to understand the meaning of a rhombus as a synthesis of isosceles or esp. equilateral triangles.

An example of two students' level 1 discussion follows:

R : What is this figure?
M9 : and M14: A rhombus
M10 : This is to say, a rhombus consists of two equilateral triangles.

M10 formulated an inaccurate definition of a rhombus after seeing the diagram, having been confused by the visual components of the rhombus on her screen, which consisted of two equilateral triangles. So this point is evidence that her formulation came as a result of misunderstanding. So this point is evidence that her formulation comes as a result of misunderstanding. She faced a cognitive obstacle that led her to a cognitive conflict when she saw the construction of the rhombus as a reflection of the isosceles triangle. Subsequently, she did not have the competence to order the two kinds of triangles and to understand the rhombus as a synthesis of two isosceles triangles.

c) A third intended activity will be to have the students build a robust construction of a rhombus. The cognitive task for the students is to connect the structure of the rhombus with the meaning of reflectional symmetry, and consequently see it as a reconfiguration (Duval, 1995) of the isosceles triangle. This case is one of many possibilities to approach to this concept.

So, they will be challenged to find ways to construct a robust construction. Furthermore, they will be able to perceive the hierarchy of the rhombus as a synthesis of isosceles or equilateral triangles. This is another point of dynamic reinvention through linking representations. According to Dina van Hiele (Fuys et al., 1984) the students will "direct their thinking activity of the students to the analysis of structure prior to the formation of associations" (p. 177).

De Villiers (1994) refers to the hierarchical classification of concepts as "a classification of a set of concepts in such a manner that the more particular concepts form subsets of the more general concepts" (p.11). The students of levels 1 or 2 are not able to form a hierarchical classification of concepts. According to Clements, Battista & Sarama (2001) this ability to classify figures hierarchically, by ordering their properties are possible only at level 3. (p. 4).

d) After the investigation process has been completed, the students will decode the image of the rhombus as a figure on screen, developing strategies of

3 The students could use only the commands "construct a parallel/ or perpendicular line" from the Construct menu which they already knew from the previous investigated activity. I limited the students to using the fewest commands possible, preferring they use only the necessary tools and the theories of geometry.

4 An oblong is a quadrilateral whose angles are all right angles, but whose sides are *not* all the same length. As Euclid defined it: *Of **quadrilateral figures**, a **square** is that which is both equilateral and right-angled; an **oblong** that which is right-angled but not equilateral;*(http://www.proofwiki.org/wiki/Definition:Quadrilateral)

5 The students will construct the figure of a rhombus as a specialization of the figure of a parallelogram they had constructed earlier on a previous screen of the software. By doing this, their knowledge of the properties of a rhombus will be built on their prerequisite knowledge of a parallelogram.

6 Many times the students avoid this special case, unless they are motivated by the researcher or the teacher to do it.

the construction of the congruent sides in the software. For the reasons mentioned above (i.e., the hierarchical classification of concepts of isosceles and equilateral triangles and consequently the hierarchical classification of the structure of a rhombus constructed as a reconfiguration of the isosceles and/or equilateral triangle) this approach is considered better than others. Moreover, the properties of the rhombus are built on the symmetry of the isosceles triangle. The knowledge of a figure's symmetry is directly connected with the defining of its properties. "Should one skip the analysis of the concept of symmetry, then one cannot expect that the pupils will arise above the already existing global structuring, because the context does not allow for an extension of the structure" (Dina van Hiele in Fuys et al, 1984, p. 160).

So, a new issue will arise: How can an isosceles triangle be constructed on screen?

This procedure has a broader aim: the understanding of the properties of the figure of the rhombus as an extension of the properties of the figure of the isosceles triangle—in other words, conceptualizing the structure of the isosceles triangle in order to cognitively structure the rhombus figure. This is to say the isosceles is a symbol in student's mind and the rhombus can be replaced with a symbol with the following attributes: "four congruent sides, congruent opposite angles, diagonals that are intersected and are perpendicular bisectors to one another" (Dina van Hiele in Fuys et l., 1984, p.207).

According to Dina van Hiele (Fuys et al., 1984) the word symbol should be interpreted as meaning "a mental substitute for a complex of undifferentiated relations that is subsequently elaborated in the pupil's mind' (p.207). By this process, the students build up the meaning of the rhombus, and the rhombus will acquire the symbolic character.

On the other hand, the synthesis of the rhombus as reflection of the isosceles leads to the analysis and synthesis of the process which is in accordance to Duval (2006) contributes "[to the general development of their capacities of reasoning, analysis and visualization]"(p.105)

Redesign process : At this point I introduced a parametrical segment (e.g, labelled CD) (see Patsiomitou, 2008, 2009). Let me explain, giving an example of the research process.

Most students -although they worked in different groups- tried to construct an isosceles triangle using the procedures they use in the static means. First, they constructed a segment AB and then they tried to construct two circles with equal radii. This process is not easy in the dynamic geometry environment, because it cannot be achieved through measurement as one can do in static means. So, they have to find another way to construct the congruent radius of the circles, or the congruent circles. The students faced many difficulties trying to interpret their mental representations.

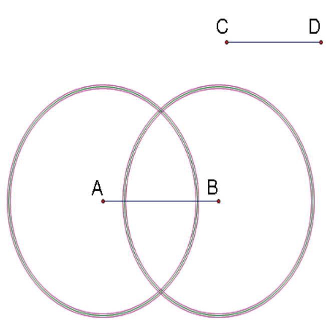


Fig. 8

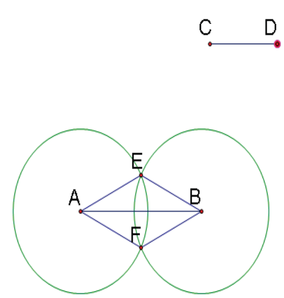


Fig. 9

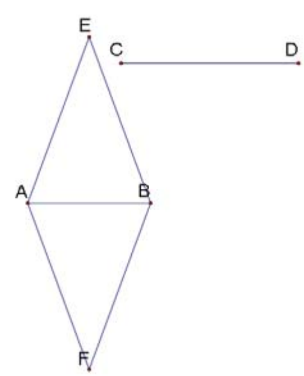


Fig. 10

Transforming Linking Visual Active Representations of a rhombus construction

I dragged the parametrical segment CD until it would become greater than the half of the segment AB (Fig. 8). Therefore, by using the parametrical segment to construct the circles and then by dragging its end points, the students would have the opportunity to link the process with the theory of geometry. The introduction of the parametric tool helped students (especially of level 1) understand the process. According to Dina van Hiele (Fuys et al., 1984) "reflection upon the manipulation of material objects, by taking the relations between those shapes as an object of study, can lead to geometry" (p.218)

Here is an example of the research process:

R : How can we construct an isosceles triangle using a compass?

M14: We use a radius (for the construction of the circle) greater than half of the segment AB.

M9 : No! We use a radius less than half of the segment AB. Then M14 dragged the endpoint D of the parametric segment. She observed the transformations of the tool in the diagram.

M9 : Oh! The radius is greater than half of the segment AB, so you were right. It depends on the distance of point C from segment AB.

M9 has understood the process of constructing an isosceles triangle during her participation in class. M9 faced a conceptual obstacle that led her to a cognitive conflict. The dragging of the parametrical tool in order to become greater than half of the segment led M9 to reformulate the definition of the constructive process of the isosceles triangle. Concretely, M9 first defined the isosceles triangle as a figure "which is constructed with a radius less than half of the segment" and after the interaction with the parametric tool as "a figure which is constructed with a radius greater than half of the segment." Subsequently, through the process and in response to instrumental genesis, she constructed an instrumented action scheme that resulted in the construction of the concept-in-action.

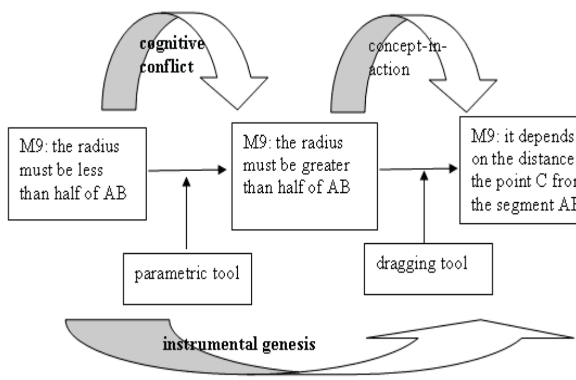


Fig. 11a

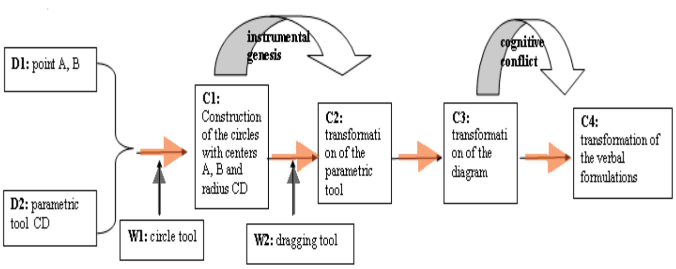


Fig. 11b

In the figure above, through a pseudo-Toulmin's model, I have represented the process through which the students were led towards a transformation of their verbal formulations. These formulations are a result of the cognitive conflicts procured during the transformation of the dynamic diagram. Both the parametric tool and the dragging tool are intervened/ intertwined into the transformation of the verbal formulations. In the diagram, the points of the students' dialogue are pointed out where the interaction with the tools becomes crucial.

4. *Problem 4 : Construction of a square. Construct a square with a free procedure.*

Design process : With the construction of a square the investigation of the students' understanding of the hierarchical relationship is aimed at (a) a specialized rectangle with additional properties (e.g., the congruency of its sides) and (b) a specialized rhombus with additional properties (e.g., the congruency of its angles). A robust construction of a square can be achieved through many alternative procedures, meaning the students "must analyze the spatial aspects of the [square] and reflect on how they can build it from [their] components" (Clements, Battista & Sarama, 2001, p. 6). A basic component is the square's congruent sides, which is a common property with a rhombus (or a square's congruent angles which is a common property with a rectangle). A main question is how the students could combine these two important processes. With these processes, the students will construct the properties of the square regarding its angles and sides, that is, regarding the figure's primary properties.

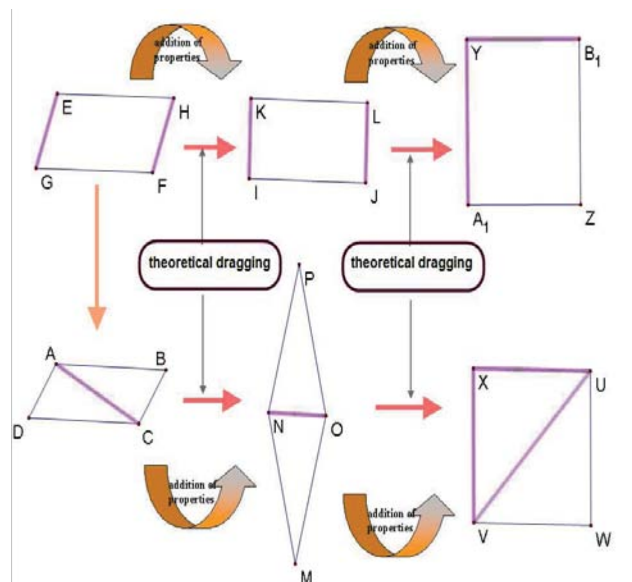


Fig.12. Linking Visual Active Representations of the first phase

Redesign process : This is a good point for the students to be introduced to the rotation[7] of a segment. This means that the students interact with an intermediary representation before seeing the final rotation of the object on screen. "Rotations play a fundamental role in forming geometric figures"(Clements, Battista & Sarama, 2001, p.55). During instrumental genesis the students will construct an instrumented action scheme of the rotation and the concepts-in-action of the congruency and perpendicularity (Patsiomitou, 2008a, 2010). Consequently, "the students will be [able] to focus on the concept of rotation, rather than focusing on the shape being rotated, [meaning] they can directly interact with a visual representation of rotation" (Sedig et al., 2001; de Souza & Sedig, 2001 in Sedig & Sumner, 2006, p.35)

ii. *A brief discussion of the first phase :* The procedure of the construction of parallelograms can be accomplished through the building of linking visual active representations. In the Figure 12 above, we can see the linking visual active representations of the first phase. Dragging the parallelogram theoretically, we can shape a "soft" rectangle, and by dragging the rectangle theoretically, we can shape a "soft" square. If we construct a diagonal in the parallelogram, we can drag it theoretically and shape a rhombus and then a square by analyzing the figure as two subfigures. Consequently, the theoretical dragging is a non-linguistic warrant to students' perceptions. For the construction of the rectangle, the parallelogram is the data, and then the rectangle will become the data for the construction of the square. By this way the students become able to perceptually form a hierarchy of the figures through linking representations.

The accomplishment of the first phase evoked a crucial issue for me: Can students use the figures' secondary properties to accomplish the construction of a parallelogram? By secondary properties are meant the properties of the figure's diagonals, which relate to the symmetry of the shape. This is in accordance with what Dina van Hiele (Fuys et al, 1984) argues, that "a student proves he possesses the structure of the analysis when he shows that he can manipulate the organizing principles. One of those organizing principles is symmetry" (p.184). For this, it is very important that the students follow the second phase.

b) ***Phase B: Investigating and building figures through symmetry***

i. Instructional design process: In this phase the notion of symmetry is introduced by using the transformations of the rotation and reflection of the software. The recognition/understanding of the symmetry of geometrical objects is the fundamental aim of this study, in accordance with van Hiele's theory, as mentioned above (see 4.1.1.3).

- The transformations of the rotation results in the construction of a symmetrical by center object in the software, by interacting with an intermediary representation. This means that the rotational symmetry is a rotation of the object for the specialized case for an angle of 180o.
- The transformation of the reflection results in the construction of a symmetrical by axis object in the software. This means that the reflection of an object (i.e., a segment or an angle) in the software and its symmetrical by axis object in a paper-pencil activity could provide perceptually the same result. Consequently, the reflection line could be interpreted as the axis of symmetry of the objects (the original object and the reflected object).

The aim of this procedure is "to introduce students to geometric transformations and to help them construct cognitive 'building blocks', such as mental rotation of shapes, that are important in dealing with spatial problems. Concepts of congruence and symmetry are explicitly addressed here as well" (Clement, Battista & Sarama, 2001, p. 12).

I separated the second phase into four subphases :

- Part B1. The recognition-visualization part of the second phase
- Part B2. The perceptually componential analysis part of the second phase
- Part B3. The informal componential analysis part of the second phase
- Part B4. The formal componential analysis part of the second phase.

1. *Part B1. The recognition-visualisation part of the second phase Problem: Reflect point A (on a given line l) in order to construct its image, point A΄. Imagine that point A will approach the reflection line l (don't use the dragging mode of the software). Describe the movement of point A΄. Will it approach or move away from the reflection line? Then drag point A until it approaches the reflection line and check your previous formulation. What do you observe? Do you have to revise your previous statement? Give reasons.*

Design process : The reflection of a point

In this stage I have re-adapted Teppo's (1991) activities. Teppo (1991) adapted the activities and used in the phases of learning, "from suggestions in 'Structure and Insight'"(van Hiele, 1986)" (p. 212). The task of the activities, being investigated by the students in the DGS environment, was a formulation that has been affected by the reflection of the dynamic object. I shall explain in details the complete process in the following paragraphs.

[7] Let me describe how to rotate a point A: First, you have to select a point O to act as the center for rotation, then select the object (s) you wish to rotate, and finally choose the rotation command from the Transform menu. The Rotate dialog box appears, which gives the students the opportunity to write the angle they want to rotate the object(s).

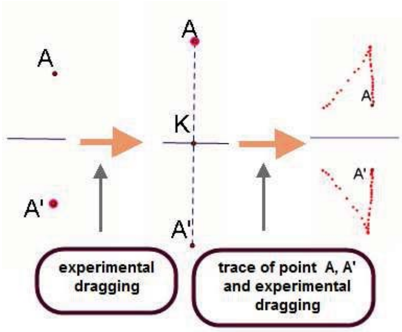


Fig. 13. a, b, c

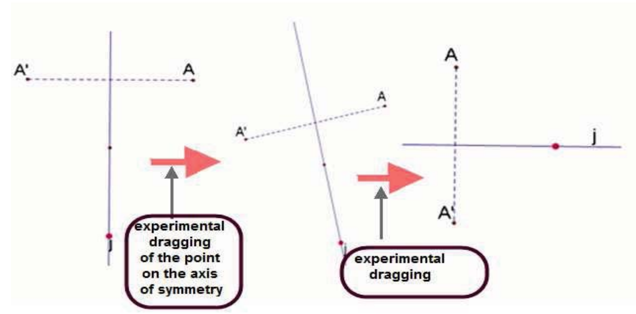


Fig. 14.

a) The direct manipulation of the hide/show action button will appear the construction of a point and its reflected point (Fig. 13). The reflection of a point is a "child's" point of view of the original point and is dependent on it. The students will drag the point or the reflected point in order to visualize the relationship of their distances from the reflection line. The dragging of the points will lead students to visualize that the points are symmetrical by axis of symmetry the reflection line.

Consequently, any action on the original object leads to the equivalent action on the image, i.e., the dependent object. This means that the students will be led to a "visual explicitness of encoded information and facilitating perception of […] transformations inherent in the VMR" (Sedig & Sumner, 2006, p. 14). Through instrumental genesis, the students will construct an instrumented action scheme of the reflection and the concept-in-action of the congruency of distances of the points A, A ´and the reflection line (Fig. 13).

At this point, a main question arise: Do students understand that the congruency of distances mentioned above holds fast for every point and vice versa on the reflection line? In other words, are the students able to conceive the generalization of the concept of congruency between the reflected points and the axis of symmetry?

An artefact that can affect the perception of generalization is the trace command. According to Jahn (2002), the "trace command emphasises a dynamic interpretation of the representation of a trajectory of a point" (p. 79) as "a set of pixels highlighted on the screen […] allows the user to instruct certain objects on screen to leave a trace when they are moved, either manually using the mouse or through the use of the 'Animation' tool."

By tracing the original point and the reflected point, the students are able to investigate the properties of the reflection in a general form (Fig. 13c). This means that through this process the students have the opportunity to visualize the congruency of the distances of the original point (and the reflected point) from the axis, moreover, the perpendicularity that is verified visually for a point on screen, theoretically for an infinite number of points.

By dragging the point of the axis of symmetry (see figure 14) in order to change the orientation of the axis of symmetry, the students will understand that the properties of the transformed objects remain stable. This is complex transformation, meaning a rotation of the reflected points. The objects change their orientation, and the challenge is for the students to grasp the meanings "through motion," which helps them generalize the concepts they have conceived before and develop inductive reasoning.

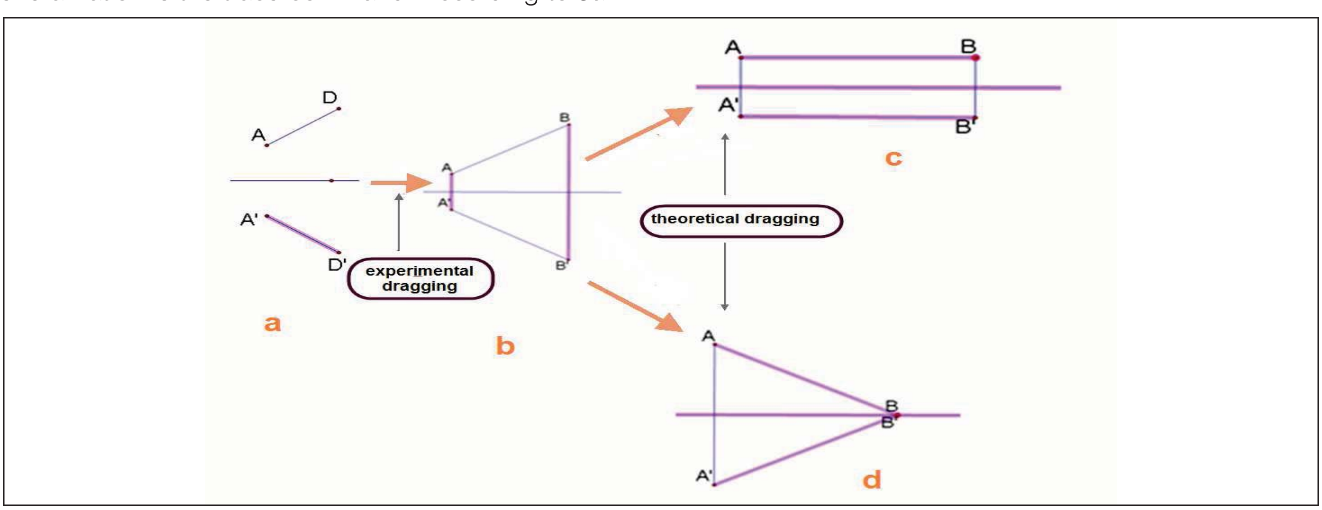


Fig. 15. Linking Visual Active Representations of the B2 part of the second phase

Design process : The reflection of a segment

The students will drag the endpoint of the segment in order to investigate how the orientation of the segment or its image will be modified, as well as the distance of its endpoints from the axis of symmetry (fig. 15 a, b). By joining the endpoints with their images with segments the students will visualize the configuration of different quadrilaterals such as isosceles trapezium, rectangles or squares. Moreover it will be investigated if the reflection line will be coincided by the students with the meaning of the perpendicular bisector. This stage has a few important parts which are described below.

Figure 15 b : The students will visualize through experimental dragging to several types of quadrilaterals (e.g., a trapezium and its properties). This is a crucial point for the research, because the figure is componentially analyzed in congruent sides and sub-figures of the shape. The questions addressed to the students are as follows: What figures do you observe? What are their properties?

Figure 15 c : The students will drag the end point of the segment AB so that it becomes parallel with the reflection line. The figure is transformed into a rectangle as a synthesis of its two componential parts (the two sub-rectangles shaped on screen). Moreover, the reflection line is the perpendicular bisector of the vertical sides of the rectangle. By dragging the end point of the segment, the students will be able to see several types of rectangles formed on screen. Furthermore, this is a good point for the students to visualize a square like a rectangle whose sides become congruent.

Figure 15 d : The students will drag the end point of the segment AB so that it will become a point on the reflection line. It is crucial for the students to recognize the isosceles triangle even if it appears in a different orientation on screen than the students usually know. The students have to recognize an isosceles triangle's componential parts formed by the reflection line (the two right triangles) and that the reflection line is the perpendicular bisector of the triangle (or the formed rectangles).

I investigated if the students developed the competency to perceptually recognize the components of the shaped figure on screen. I will give an example of the research process. I dragged the endpoint of the segment AB until it touch the reflection line (Fig. 15 d).

R : What is this figure?

M1: This is a right triangle and this is an isosceles triangle.

R : And can you explain why these are intersected on the reflection line?

M1: Perhaps because the software kept this triangle as an isosceles.

M1's (van Hiele level 1) expression ("because the software kept this triangle as an isosceles") could be reformulated as "the objects of the software preserve the properties for which they are constructed, which results the congruency of the segments and then that the triangle remains isosceles." M1 recognized the subfigures in which an isosceles triangle is separated from the reflection line, although the isosceles had an unusual orientation on the screen. Subsequently, M1 has developed the competency to perceptually recognize the components by which the figure is analyzed.

2. *Part B2. The perceptually componential analysis part of the second phase*

Problem : Construct an axis of symmetry of rectangle.

Design process : The construction of rectangles' axes of symmetry

The students will face difficulties in understanding the meaning of axis of symmetry and how it differs from rotational symmetry, which is expressed with the misunderstanding of the roles that the secondary elements (for example, the medians of a triangle or the diagonals of a rectangle) play in the figures' symmetry. Another point is students' difficulty in distinguishing the difference between the meanings of "symmetry of an object with regard to an axis of symmetry" and the meaning of "symmetry lines of the shape." Symmetry lines are those lines which the construction of the symmetrical point for any point on the figure leave the figure unchanged. The construction of the diagonals of the rectangle as rectangle's axes of symmetry is a commonly known misunderstanding faced by many students (Panaoura et al., 2009, p. 46).

There are researchers who give evidence that such misconceptions have even appeared to preservice teachers. According to Son (2006)

"It was found that a large portion of pre-service teachers had lack of content knowledge of reflective symmetry. A large portion of preservice teachers had misconception of reflective symmetry. They misunderstood that the parallelogram had lines of symmetry. They confused symmetry and rotation. When they were asked to explain how to perform reflection, over half of preservice teachers relied on the procedural knowledge of reflective symmetry such as folding rather than focused on the properties of reflective symmetry [...]. It is revealed that many prospective teachers confused the property of reflection and those of rotation" (and had tendency to rely on the procedural aspects of reflective symmetry when using teaching strategies) (pp.149-150).

Through the current process the students pursue conquering the cognitive tasks

- Correlating the construction process with the investigational part of the current phase and overcoming the conceptual obstacles correlated with the meaning of the axis of symmetry with the construction of the diagonals of the figure.

- Perceptually understanding the axis of symmetry as a result of the connection of the midpoints of the opposite sides of the shape and consequently to construct the meaning of the midpoint-parallel line. In other words to dynamically reinvent a rule "the segments that join the midpoints of the opposite sides of the rectangles are its symmetry lines".
- Equating the two processes and consequently connecting the primary and the secondary properties of the shape.
- Defining the axis of symmetry of the rectangle and constructing a definition of the rectangle based on the definition of the axis of symmetry.
- Investigating and reasoning whether the axis of symmetry are perpendicular

An example of the research process includes the following discussion:

M_7: let's find the rectangle's axes of symmetry. I know… I mean, we have to join the diagonals…
M_8: What for?
M_7: It will pass from this point (the intersection point of the diagonals), it must be parallel here (and points to JI) and pass from here (points to intersection point of the diagonals)… and be vertical here (and shows towards HI)
R: Are GI and HG the axes of symmetry?
M_7: No!
M_7: Let's join the midpoints.

Therefore, we have a theoretical construct derived through interaction with the on-screen diagram. M7 related the reflection of the objects with the symmetry by axis, meaning that she related procedures with meanings. Meaning the linking representations that she created during the process helped her to correlate the primary and the secondary properties of the figure, meaning the notion of perpendicularity to that of parallelism. In this way, the student assimilated that the interparallels are perpendicular to the sides.

Consequently, the construction of the meaning, "the axes of symmetry are the lines that join the midpoints of the sides of the figure," is a result of this process.

Construction of the axes of symmetry of a rhombus

Problem: Construct the axes of symmetry of rhombus. Then join the midpoints of the opposite sides with a segment and explain why it is an axis of symmetry or not. Then, drag the vertex of the rhombus to form a square.

Design process : The construction of rhombus' axes of symmetry

Most students intuitively know that the axes of symmetry of a rhombus are its diagonals. This is a crucial point for the research process because the students have to overcome a cognitive obstacle: The segment that joins the midpoints of the opposite sides of the rhombus is not an axis of symmetry because this line is not perpendicular to the sides of the rhombus. By using experimental dragging they will perceptually understand that the axes of symmetry of the rhombus do not follow the rule that the rectangle does. I will give an example of the research process.

I asked the students to construct the midpoints of the opposite sides and then to answer the question, "What is the segment that joins the opposite midpoints of an axis of symmetry of the rhombus?" They then had to explain their answers.

R : Is the segment OP an axis of symmetry?
M8, M13: Yes
M7 : Yes, this is the midpoint
M7 : Oh no! It is not because this angle is not right!!

M7 faced a cognitive conflict when she visually does not verify the property of the perpendicularity at the interparallel line of the rhombus. She has previously correlated the interparallel line of the rectangle with the meaning of symmetry line of the figure. Subsequently, she was leading to accommodate the cognitive scheme she has constructed for the meaning of axis of symmetry for the case of rhombus. This means that M7 has acquired "an increasing ability and inclination to account for the spatial structure of shapes by analyzing their parts and how the parts are related" (Battista, 2007, p. 851). This is a result of the mental connection of representations at different points of the research process.

Consequently, the linking representations led the student to cognitive conflicts and prompted her to develop her thinking processes, mediating to the decoding of her mental image to an iconic representation and then to a verbal one.

Construction of the axis of symmetry of a square

Design process : The construction of a square' axes of symmetry

The students have to recognize/realize that the square concentrates all the properties that the previous shapes did, with regard to its symmetry lines. This means the segment that joins the midpoints of the opposite sides of the square is a symmetry line, as are its diagonals, so the square concentrates all the properties of the rhombus and the rectangle with regard to symmetry lines. This means that the students can give hierarchy to the square as a rhombus or a rectangle and define it from its properties from the lines of symmetry.

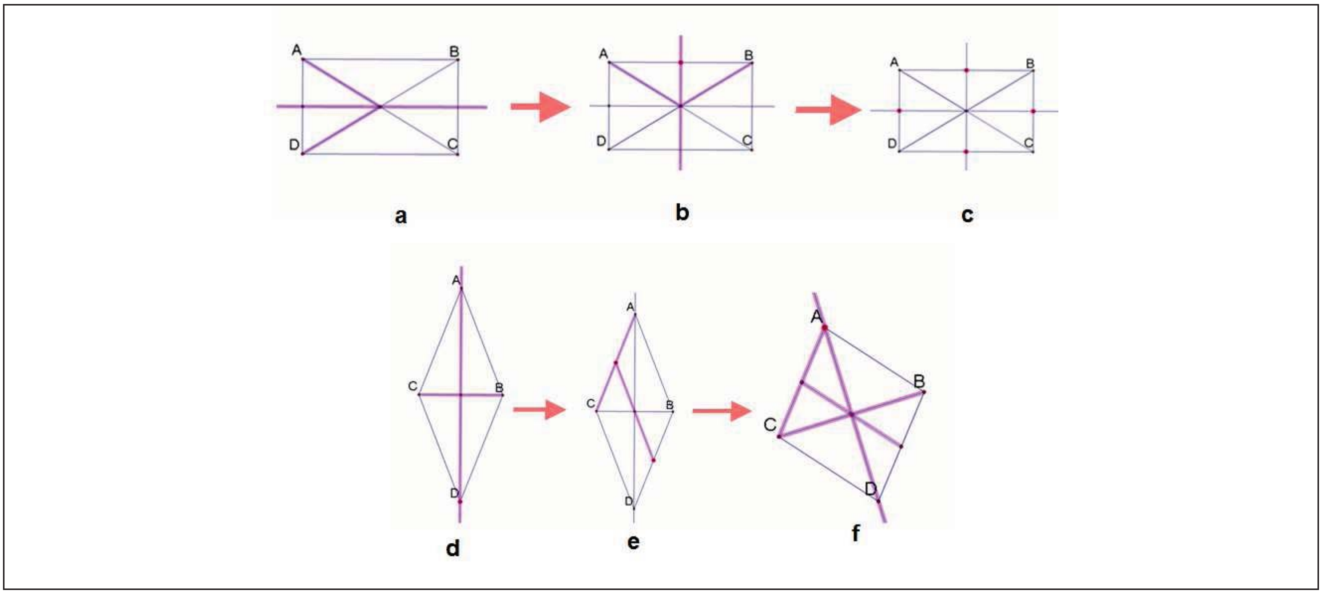


Fig.16. Linking Visual Active Representations (or non) of the B2 subphase

In a brief discussion of the part B2 of the second phase, I observed that the students linked in their minds the representations that helped them answer at the next level. In the figure above, we are able to see the linking visual representations between the phases of the same construction, or between the constructions of different parts of the same phase. For example the steps a, b, c of figure 16 are linking representations of the construction of the rectangle as they link the different procedural aspects of the same process. The steps c and d of Figure 15 are linking representations of the steps a, b and c of Figure 16 because they mentally link the properties of the construction steps. The steps a and e of Figure 16 are linking constructional steps of an inquiry process but they are not linking representations of the figures because they do not mentally link the processes or they lead to a cognitive conflict. The steps a, d and f of Figure 16 are linking representations because they link the hierarchy of the figures through the properties of their axes of symmetry.

3. *Part B3. The informal componential analysis part of the second phase*

Redesign process : The investigation of the meaning of rotational symmetry

The students' cognitive conflicts led me to redirect my study in order to include the investigation of the meaning of rotational symmetry. The students were confused about the two meanings and most students believed that the rotational symmetry of a point can be defined as a reflectional symmetry of the point.

The task was for the students to build on their prior knowledge, on what they have learned through their participation in class, so I prompted them to rotate the point by joining point A with point O and then to follow the instructions, which means they had to transfer their knowledge of how a point can be rotated in static means in the DGS environment.

In order to facilitate the process, I created a 'custom tool' that could apply the procedure of the rotation of a point by 180o, appearing only as the final step of the rotation process (meaning the students could not see the entire intermediary steps of the rotation process) (see Patsiomitou & Emvalotis, 2009, 2010).

This means that the students can see on screen the segment that joins point B with point O and also the segment OB´ (Fig. 19). Consequently, the result of this procedure is the same as that in which students used the rotation command. Students using the rotation command can interact with the intermediary representation through which they can define the rotation angle, meaning they interact with the linking representations that occur on screen. But the 'custom tool' operates in an abstract way and displays only the final result. According to Jackiw (personal e-mail communication with Nicholas Jackiw, September, 29, 2005) "scripts [or custom tools] represent an abstraction of your own work or process, and thus using them as "abstract tools" requires a level more advanced or sophisticated a conceptualization than using "literal" tools like the compass and straighetge". In this way, this "custom tool" operates as a developmental indicator of a student's understanding and of his/her cognitive growth, as there is a need for the student to understand the tool's hidden principle.

Redesign process : The example and the counter-example of custom tool's use

The difficulties that arose from the use of the custom tool made me use an example and a counterexample of its use. By example I mean, where

Global Journal of Computer Science and Technology Volume XII Issue VI Version I March 2012

the "custom tool" is helpful is in understanding that the rotation of every point of the circumference of a circle on its center (rotation of the circle around its center) results in the circumference of the same circle. By counter-example I mean that the rotation of an equilateral triangle at the intersection point of the perpendicular bisectors results in a different equilateral triangle (rotation by 180° of the original at the intersection point of the perpendicular bisectors).

The example: I asked the students to rotate the circle around its center by asking, What is the symmetrical figure of a circle by its canter ?

The counter example: The intersection point of the perpendicular bisectors of an equilateral triangle is not the centre of symmetry of the triangle. I will give an excerpt of the research process

The example:

M1: He places a point A on the circle and then applies the custom tool to point A and point F.

R : What is the symmetrical of point A?

M1: This is (he points out segment OA´) He tries the process again and again for several points on the circle. Then he constructs the symmetrical point of point H using the custom tool. The counter-example:

R : Is the point O the center of symmetry of the figure of the equilateral triangle?

M1: In order for point O to be the center of symmetry it would be this segment congruent with this segment.

R : What are these segments?

M1: AO=OE

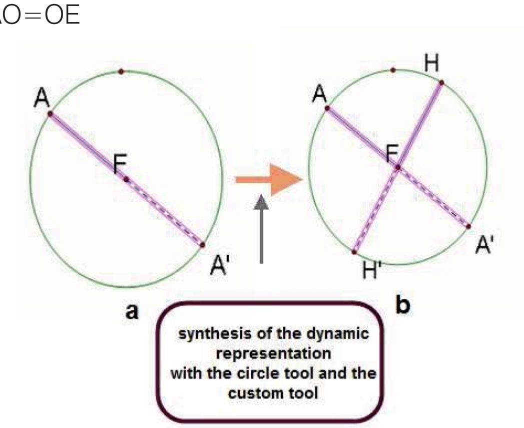


*Fig. 17.*Example of the use of the custom tool

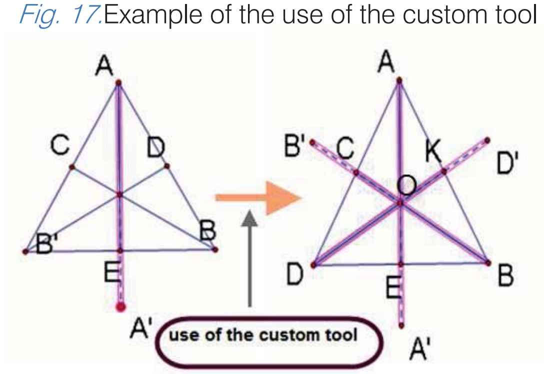


Fig 18. Counter–example of the use of the custom tool

Although the way I asked the question might be more likely to trigger a "no" without any thinking, M1 verbally decoded the iconic information, based on his visual perception and on mental transformations of visual data comparison. He has acquired "an increasing ability to understand and apply formal geometric concepts in analyzing relationships between parts of shapes" (Battista, 2007, p.852).

Redesign process : The construction of the structure of the bisected diagonals

The students will construct the image of the segment CD by rotating it by 180° around H. There are several options. From an instrumental genesis perspective, the students can construct an instrumented action scheme by using the custom tool. Moreover they will be able to construct the meaning the "diagonals [of a parallelogram] are bisected /dichotomized". According to Drijvers & Trouche (2008)

The difference between elementary usage schemes and higher–order instrumented action schemes is not always obvious. Sometimes, it is merely a matter of the level of the user and the level of observation: what at first may seem an instrumented action scheme for a particular user, may later act as a building block in the genesis of a higher-order scheme. […] a utilization scheme involves an interplay between acting and thinking, and that it integrates machine techniques and mental concepts […] the conceptual part of utilization schemes, includes both mathematical objects and insight into the 'mathematics of the machine'(p. 372)

By using the custom tool twice, with the second application point at the symmetry center O, students will lead to the construction of two segments that have the same midpoint. Consequently, the meaning of "diagonals are bisected /dichotomized" can be constructed by the students through the use of the custom tool (Fig. 20).

So they will construct a "higher [secondary] –order usage scheme" (Drijvers & Trouche, 2008, p.371). By dragging the points, they can visualize a parallelogram and that the structure of the intersected dichotomized diagonals of any parallelogram shaped on screen are unmodified. In this way, the students can construct the structure of the parallelogram from its symmetry properties and the symbol character of the parallelogram is accomplished with a secondary property. The construction of the rotational symmetrical triangle is an important part of the whole activity. It is crucial for the students to recognize the parallelograms within a complex figure and to formulate their arguments.

Consequently the procedure will help the students to recognize the figure of its properties, meaning the figure will acquire the signal character. The images in Figure 20, 21 are linking representations of the higher–order utilization schemes. This means that these representations are linked.

- Structurally as the dragging of any point does not modify the structure of the construction.
- Conceptually through the meaning of the symmetry by center and the meaning of the intersected bisected diagonals.

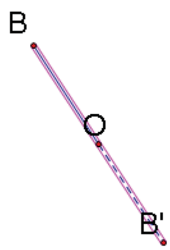


Fig.19

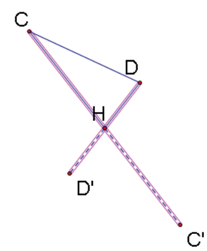


Fig. 20

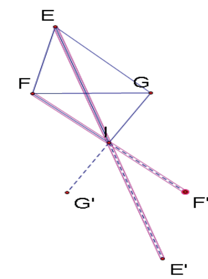


Fig. 21

Furthermore it will be investigated whether the students are able to understand "the objects' double status" (Duval, personal e-mail communication with Prof. Duval, August 3, 2010). This means to interpret any object (for example a point or a side) as being an element of the triangle or the parallelogram that can be formed.

I shall give an example of the research process. M4 faces an instrumental obstacle, because the extension of the segment cannot be made as a straight line as is the case with the ruler in static means (Fig. 22). Consequently, it is the process that pushes her to develop her decoding ability of mental and verbal representation to an iconic one onscreen. This leads to a cognitive conflict and the dynamic reinvention of a procedure to accomplish the construction of the symmetrical object.

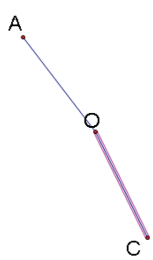


Fig. 22

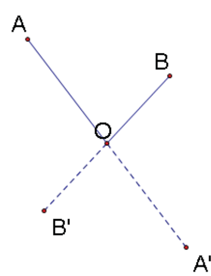


Fig. 23

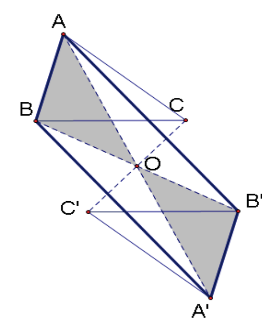


Fig. 24

R : How can we construct the symmetry by center of point A by point O?

M4 : We can extend the segment OA to segment OA´ congruent to OA.

M4 selects the segment tool and tries to construct the extension of segment OA, but she faces an instrumental obstacle, as she tries to apply a process used in paper and pencil construction. Then she tries the custom tool in order to construct the rotational symmetry of points A, B.

R : What figure is this?

M4 : A quadrilateral…oh! a parallelogram. (surprized)

R : Why?

M4 : Because its …diagonals are bisected.

I was expecting the answer "two lines intersecting at the midpoint", but M4 saw a parallelogram onscreen, although it was not completed. One possible interpretation is that the construction of the rotational symmetry of point B by center O results in the construction of the intersected segments with common point O, meaning the structure of the diagonals of a parallelogram.

So, M4 recognized the parallelogram on screen from the structure of its bisected diagonals. Dragging the construction from a point-vertex, the properties remained stable, meaning point O remained the midpoint of both the segments, as well as the points A, A´, and B, B´, thus preserving the property of the symmetrical objects. M4 was able -through the dynamic diagram- to recognize the figure and to verbalize in formal language using the criterion of the parallelogram (i.e. if the diagonals of a quadrilateral have the same midpoint then the quadrilateral is a parallelogram or if the diagonals of a quadrilateral bisect each other then quadrilateral is a parallelogram). Subsequently, the student was able -by using the custom tool- to transform an iconic representation into a verbal one through mental transformations.

R : What are these triangles?
M3 : They are congruent?
R : How is this occurred?
M4 : From the parallel lines.
R : Where are the parallel lines?
M4 : The sides are parallels because they are parallelograms.
R : What are the parallelograms?

I was surprised. M4 named all the parallelograms by mentally joining the segments of the figures in order to answer my question. She recognized the parallel lines and the structure of the bisected diagonals—in other words, the parallelogram acquired its signal character. The student saw the parallelograms, meaning she acquired the insight in order to dynamic reinvent the solution to the problem. Consequently, by linking representations and her mental transformations, she acquired the competence to structurally analyze the figure. She also gave the segments AB, AC a double status: (1) as sides of the triangle ABO and (2) as sides of the formed parallelogram ABA΄B΄.

4. *The formal componential analysis part of the second phase*

Design process : Construction of a parallelogram

The aim of this part of the third phase is for the students to construct a parallelogram with their starting point being their knowledge of the symmetry of the figure. The students will construct the figures with the prerequisite that "a specific criterion of validation for the solution of a construction problem: a solution is valid if and only if it is not to mess it up by dragging "(Jones, 2000, p. 58 in Battista , 2008, p. 353).

It will be investigated whether the figures have acquired the signal character and if the students can justify their procedures theoretically. Moreover, the synthesis of the tools that lead the students to a valid solution or to trial and error will be investigated.

Van Dormolen (1977, p.27) in his article "Learning to understand what giving a proof really means" argues:

When someone wants to solve a mathematical problem, he usually will not be able to follow a strictly deductive reasoning from the start. As a rule he begins with a more or less disorderly period of trial and error in which he tries to get a grip on the problem. After this has been successful, he will proceed to try and put his solution into a tidy form. (p.27)

Consequently, the students construct the parallelograms based on the figures' properties related to the axes of symmetry or center of rotational symmetry, meaning that they might have deduced in the second and third phases. According to Whiteley & Moshé (2005)

Once you start thinking of quadrilaterals in terms of their symmetries, you will find new ways of constructing them in Geometer's Sketchpad. Rather than using the "construct" menu, it is of more benefit to encourage students to use the "transform" menu. Emphasizing the "transform" menu in GSP can serve as a way to develop and reinforce students' transformation skills. Think about how you can construct a square using the "transform" menu. Remembering symmetries of quadrilaterals and using them to sketch the quadrilaterals will facilitate better understanding of symmetries and how essential they are in geometry. (http://www.docstoc.com/docs/17713922/Exploring-the-Parallelogram-through-Symmetry, p.4)

The students will construct the figure by taking into account the structure of its diagonals. In this current phase, it is crucial for the students to recall the properties of the figure's diagonals that were investigated in the previous facets of the research process by mentally linking the reverse representations in this procedure. This phase is very crucial for the students to acquire the ability to replace a figure with a set of properties that represent it and from these properties to construct the figure. In other words, the figure will acquire the signal character. This is a very complex process since the students must have both conceptual and procedural competence, meaning the competence to instrumentally decode their mental representations of a set of properties with actions through the use of tools. This means, for example, to interpret the congruency with the circle tool and simultaneously bisect with the custom tool.

Furthermore, for them to construct the hierarchical categorization and definition of figures through their symmetrical properties and in accordance to their understanding. According to Fujita & Jones (2007) "the hierarchical classification of quadrilaterals is difficult because it requires logical deduction, together with suitable interactions between concepts and images" (Fujita & Jones, 2007, p.12). Another important aspect is the development of their cognitive structures (McDonald, 1989, p.426).

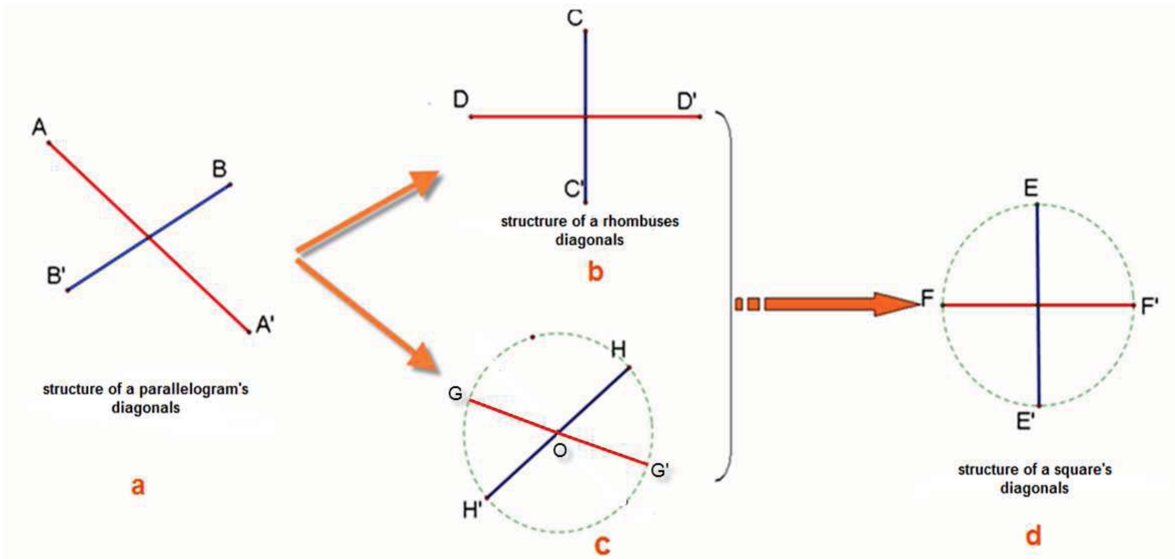


Fig. 25. Linking Visual Active Representations of parallelogram's diagonals

In the Figure 25 above we are able to observe the linking representations of the diagonals of different types of parallelograms. Dragging theoretically the endpoint of the diagonals of the parallelogram in order these to acquire the property of the perpendicularity leads to the structure of the rhombus diagonals (or a square's diagonals). Dragging theoretically the endpoint of the diagonals of the parallelogram in order these to acquire the property of the congruency leads to the structure of the rectangle's diagonals.

The construction of two arbitrary diameters in a circle (i.e. the diagonals are not perpendicular to one another) leads to the structure of the diagonals of a rectangle. The construction of two diameters perpendicular to one another in a circle leads to the structure of the square's diagonals. In this way conceptually and procedurally linking representations are created. Simultaneously, the representations of the Figures 16b, c, d, f, 17b and 20 of the previous phase are linked with the representations above with a reversion of the procedure.

Subsequently, this learning path can lead to the development of an abstract way of thought through the development of linking representations in student's mind.

I will give an example of the research process.

R : What is this figure?

M4 : A square.

M3 : It is not a square (dragging point B). It is a parallelogram.

M4 : A rectangle.

R: Why?

M3 : Its diagonals are bisected.

M4 : They are congruent.

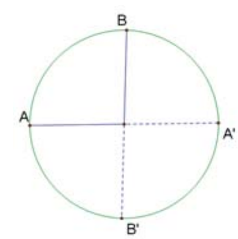


Fig .26

The construction of the rectangle is accomplished by M4 as a reversal process of the construction of the rectangle's axes of symmetry. The student faced an instrumented obstacle during the decoding of her mental image to an iconic representation on screen. She formulated the notion of the arbitrary distances, which is interpreted as actions with the construction of an arbitrary point with two degrees of freedom. Thereafter, the use of the point tool led the students into a cognitive conflict.

From M4's answer, it is concluded that the rectangle has not acquired its signal character. M4 applied the custom tool at an arbitrary point B of the circle, meaning a point with one degree of freedom. The experimental dragging of point B leads the student to a cognitive conflict and to a re - identification of the figure's properties. Consequently, the circle tool and the custom tool mediate in order for the student to dynamically reinvent the properties of the rectangle.

Subsequently, they mediate (a) the decoding of the mental representation to an iconic and then to a verbal, (b) the construction of the figure's signal character, and (3) the recognition of the double status of the figure's elements.

Through the process, the students construct the interparallel line of the rectangle as the axis of symmetry of the figure. The use of the tool mediates the construction of the meaning of the symmetric point on the perpendicular line.

ii. Observations of the second phase

The images in Figure 13 are LVARs of the construction of the meaning "every point and its symmetrical have congruent distance from the reflection line/axis of symmetry or distances (numbers) can be equal and the segments congruent". The linking representations of Figure 14 reinforces the construction and understanding of the meaning mentioned before and visually verifies (or visually proves) that the axis of symmetry remains perpendicular to the segment that joins every point with its mirror image, although its orientation has been changed.

The extension of the meaning of the perpendicular bisector into objects of 2D is a consequence of their manipulation using theoretical (or experimental) dragging in Figure 15. The axis of symmetry visually separates the figures into subfigures that preserve the congruency of their altitudes. This means that the figure of the rectangle and the figure of the isosceles triangle are structurally analyzed to subfigures due to the figure's perpendicular bisector.

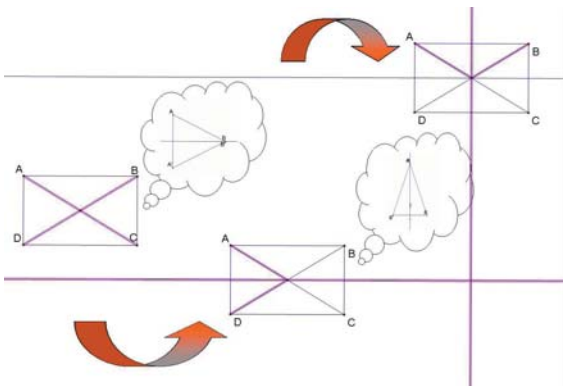


Fig.27a

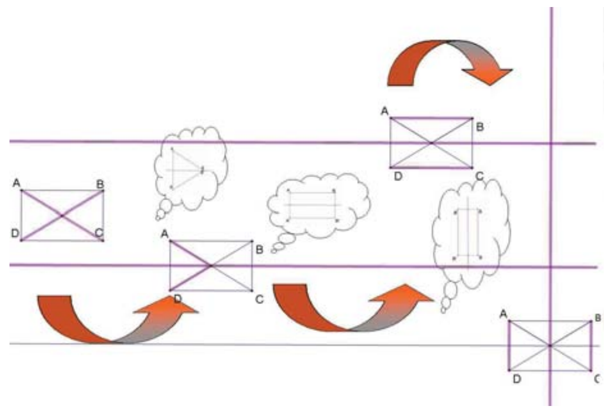


Fig.27b

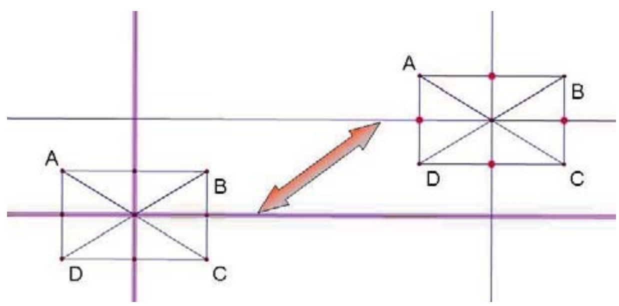


Fig.27c

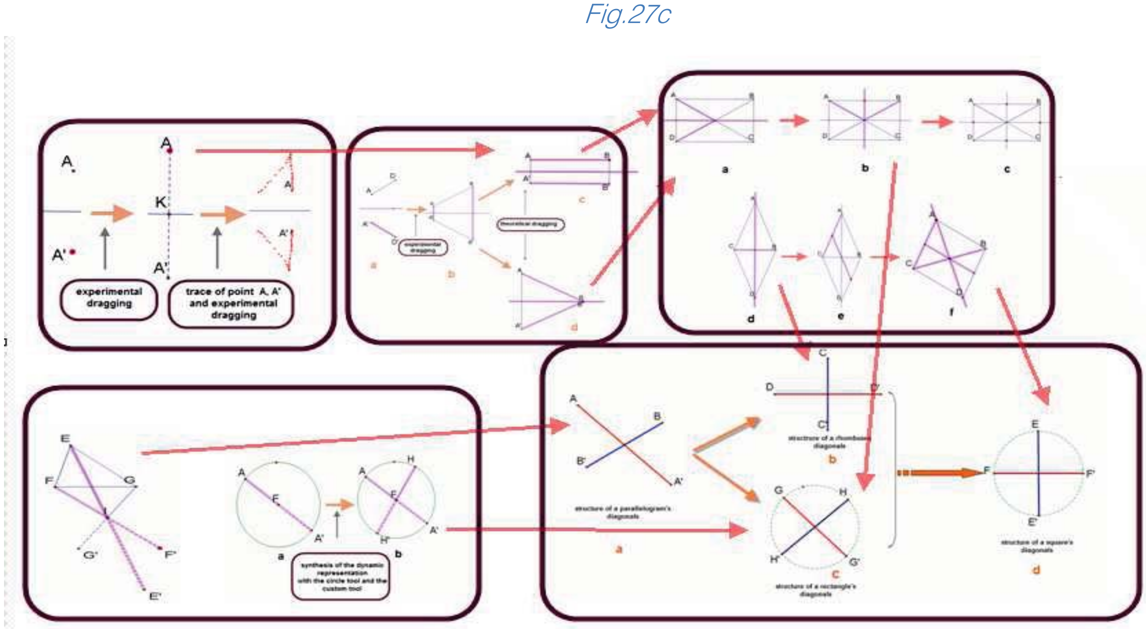


Fig. 28. Links between the geometrical objects of the second phase of the DHLP

The Figure 16 a and b form LVARs with the Figures 15c and d. The construction of their diagonals and the perpendicular bisector of the isosceles triangle shaped by the diagonals o the rectangle is a consequence of the mental connection between the representations shaped in the previous stage of the HLP. The misunderstanding and the cognitive conflicts that students will face help them to accommodate the cognitive scheme of the axis of symmetry of figures of parallelograms. The images in Figure 17 are LVARs for the construction of the meaning "the rotation of every point of the circumference of a circle around its center by 180o results on the circumference of the same circle" and the custom tool used is the 'warrant' for the understanding of the concrete meaning. The construction of the structure of the parallelograms is an abstraction process which occurs as reconceptualization step–by–step. The structure of parallelogram's diagonals procedurally is linked with the rotation of a segment. Meaning the images 19, 20 create linking representations with the image 25a. Likewise, image 16c creates a linking representation with the image 25b and image 17b creates a mental linking representation with the image 25c.

The image above (Figure 28) illustrates the geometrical objects in the concrete stages of the research process and the implied links between them are illustrated with a green arrow.

c) Phase C

The third phase follows the second phase and is in development with the last phase which concerns the investigation of problems with the LVAR modes. This phase is important for the development of the understanding of the role that the diagonals of the quadrilaterals play, the recognition of the substructures in the figures that play a significant role in the construction of proofs and the ability of the students to recognize the elements of the figures interpreting them in multiple ways. A very important problem with regard to the investigation of the web of the relationships between the properties of quadrilaterals is the Varignon problem. For any quadrilateral we can prove that the internal figure constructed by the midpoints of the sides of the external figure is a parallelogram. The students learn to prove through a procedure of the application of the midpoint-connector theorem.

Graumann (2005) in an extended and detailed description of the study of quadrilaterals classified the quadrilaterals with regard to their diagonals. In this way he distinguished the quadrilaterals into three separate categories: those whose diagonals are congruent, those whose diagonals are perpendicular and those whose diagonals are intersecting one –another at an arbitrary point. Graumann continued the classification of the quadrilaterals by adding properties into each one of the above-mentioned categories until they had been led to a specialized figure such as a square whose diagonals are congruent and perpendicular.

He has represented this classification with a figure. The internal quadrilateral is a parallelogram for every external quadrilateral. Graumann (2005) has represented this classification with a figure. I have constructed an adaptation of Graumann's figure (2005, p. 194) by constructing the internal parallelogram, joining the midpoints of the sides of the external quadrilateral.

In this way, a new classification of quadrilaterals occurs due to the different properties of the internal parallelograms. For example, the quadrilateral made from the joining of the midpoints of a quadrilateral whose diagonals are perpendiculars is a rectangle.

Also, the parallelogram which is shaped from the midpoints of the sides of the quadrilaterals whose diagonals are perpendicular and bisected to each other is also a rectangle and in addition its sides are symmetrical with regard to the diagonals of the external quadrilateral.

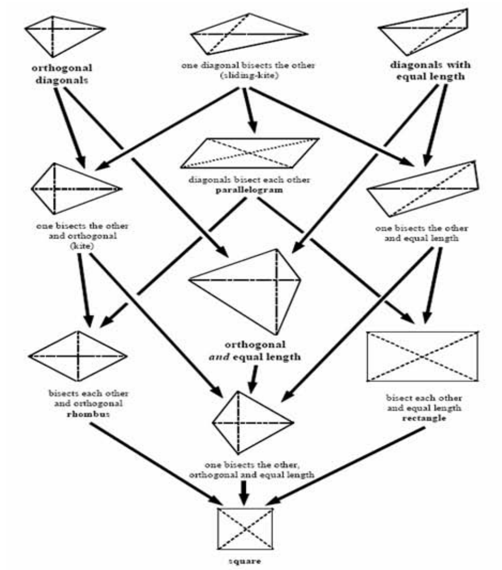


Fig. 29. Graumann's (2005, p.194) 'house of quadrilaterals' concerning diagonals.

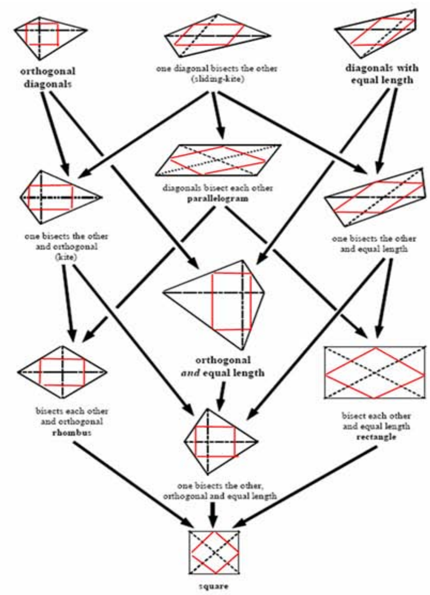


Fig. 30. An adaptation of Graumann's 'house of quadrilaterals' including the middle-quadrilateral figures.

Consequently, the classification of a quadrilateral as a rhombus is not adequate with regard to the properties of the rectangle which occurs internally. The classification of the rhombus as a quadrilateral whose diagonals are perpendiculars and are bisected accurately determines the parallelograms' shape, whose two sides are symmetrical as regards the diagonals of the kite. I have constructed a table below in which I have described the kind of parallelogram which occurs in the internal section of the quadrilateral.

External quadrilateral	Internal quadrilateral
Quadrilateral with orthogonal diagonals	**Rectangle**
Kite with one diagonal bisects the other and is orthogonal	Rectangle whose two sides are symmetrical by the orthogonal diagonal
Rhombus (each diagonal bisects the other and is orthogonal).	Rectangle whose opposite sides are symmetrical by diagonals.

It is obvious how the students were able to create LRs with the previous phases of the hypothetical learning path due to the construction of the formed quadrilaterals in the internal of the shape.

d) Phase D

The LVAR modes corresponding to the apprenticeship phases mention in the section regarding the theory of van Hiele are described as follows (for example Patsiomitou, 2008a, b, 2010):

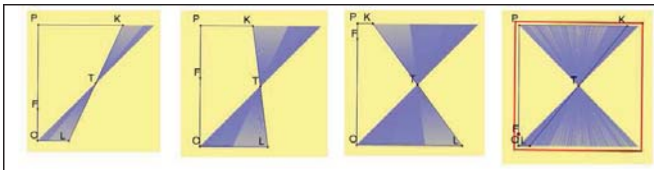
Mode A-the inquiry/information mode : In this phase of the problem, the students familiarize themselves with the field under investigation using the instantiated parts of the diagrams which lead them to discover a certain structure.

Mode B-the directed orientation mode : In concrete terms, the sequential linked constructional steps of the solution to the problem emerge step-by step.

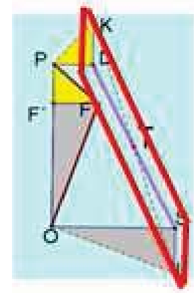
Mode C-the explicitation mode : Transformations in increasingly complex linked dynamic representations of the same phase of the problem modify the on-screen configurations simultaneously.

Mode D-the free orientation mode : Every phase in the solution can be displayed side by side on the same page of the software in an overview.

Mode E–the integration mode : Successive configurations on different pages that are linked cognitively and not necessarily constructionally, compose the solution to the problem in global terms as a series of steps.



Mode A



Mode B

For example in modes A and B:

The figures above modes A and B illustrate the bisected diagonals of a parallelogram as the point P is moving on its path (segment PO at mode A) or as the point F is moving on screen generally. Dragging point P or point F forms several figures (rectangle, square, etc.) so students are able to link the solution with their pre-existing knowledge acquired in previous phases. A crucial point is that the students constructed a second order utilization scheme for the rotation of the triangles (phase B, C, and D). So they concluded that these triangles were congruent by extending the previously constructed utilization scheme for the rotation of segments in phases A and B.

Therefore, it appears that the use of LVAR in the Sketchpad dynamic geometry environment proving process can organize the problem-solving situation and the structuring and restructuring of the user's instrumental schemes it evokes as the activity unfolds. As the LVARs' composition changes, there was a transformation of the user's verbal formulations. Consequently, the scheme of use associated with the constructed instrument changes led the participated students to pass from an empirical to a theoretical way of thinking or to students' mental transformations.

V. Discussion

The design and redesign of the DHLP as well as the results occurring from the research process, as have been reported in previous papers (for example Patsiomitou, 2008a, b, 2010, 2011; Patsiomitou & Emvalotis, 2009 b, c, 2010) led me to conclude that a student could develop abstract ways of thinking when his /her cognitive structures were linked with mental linking representations. LVARs could completely differ among students and are dependent on the student's conceptual understanding, his/her development of abilities, and thinking processes. A student can construct linking representations:

- When s/he builds a representation (for example, a figure) in order to create a stable construction, using software interaction techniques by externalizing his/her mental approach or generally by transforming an external or internal representation to another representation in the same representational system or another one.
- When s/he gets feedback from the theoretical dragging to mentally link figures' properties so that, because of the addition of properties, subsequent representations stem from earlier ones.
- When s/he transforms representations so that the subsequent representations stem from previous ones due to the addition of properties.
- When s/he links mentally the developmental procedural aspects in a process of a dynamic reinvention
- When s/he reverses the procedure in order to create the same figure in a phase of the DHLP or between phases of the same DHLP.

For this I redefine the notion of Linking Visual Active Representations below in order to include all the occasions mentioned above.

Linking Visual Active Representations are the successive building steps in a dynamic representation of a problem, the steps that are repeated in different problems or steps reversing a procedure in the same phase or between different phases of a hypothetical learning path. $LVAR_S$ reveal an increasing structural complexity by conceptually and structurally linking the transformational steps taken by the user (teacher or student) as a result of the interaction techniques provided by the software to externalize the transformational steps s/he has visualized mentally (or exist in his/her mind) or organized as a result of his/her development of thinking and understanding of geometrical concepts.

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Global Journal of Computer Science and Technology Volume XII Issue VI Version I March 2012

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