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Effects of Internal Heat Generation and Variable Viscosity on Onset of Rayleigh-Benard Convection

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Abstract - In the present study, onset of stationary Rayleigh-Benard convective instability in a fluid layer, with internal heating and thermally dependent viscosity has been investigated by means of linear stability analysis. The dependence of viscosity is assumed to be exponential. The resulting eigen value problem is solved using a regular perturbation technique with wave number *a* as a perturbation parameter. The viscosity parameter and the presence of internal heat source play a decisive role on the stability characteristics of the system. It is observed that both stabilizing and destabilizing factors can be enhanced because of the simultaneous presence of a volumetric heat source and variable viscosity so that a more precise control (suppress or augment) of thermal convective instability in a fluid layer is possible.

Key words: variable viscosity : internal heat source: Rayleigh-Benard convection.

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1. Introduction

Convective flow in a thin layer of fluid, free at the upper surface and heated from below, is of fundamental importance and becomes a prototype to a more complex configuration in experiments and industrial processes. The convective flows in a liquid layer are driven by buoyancy (Benard) forces due to temperature gradients and thermo capillary (Marangoni) forces caused by surface tension gradients and have long been studied extensively since the pioneering experimental and theoretical works of Benard (1900), Rayleigh(1916), and Pearson (1958).

The mechanism of internal heating in a flowing fluid is relevant to the thermal processing of liquid foods through ohmic heating, where the internal heat generation serves for the pasteurization/sterilization of the food Ruan et al. (2004). Other important applications of flows with internal heat generation are relative to nuclear reactors, as well as to the geophysics of the earth's mantle. In both cases, the internal heating is due to the radioactive decay. For nuclear reactors, processes of natural convection with internal heating are extremely important in the analysis of severe accident conditions. As pointed out by Generalis and Busse (2008), flows with volumetric heating are relevant for the physics of the atmosphere, in connection with the absorption of solar radiation. Due to the wide range of industrial and geophysical applications, extensive literature has been recently produced on this subject; see e.g. (Carr 2004, Carr and Putter 2003, Hill 2004, Straughan 2008, Straughan and Walker 1996, Tse and Chasnov 1998, and Zhang and Schubert 20020).

Physically, all fluids posses a temperature-dependent viscosity, whose effects are important on the onset of Rayleigh-Benard convection.(Torrance and Turcotte 1971, Booker 1976, Booker and Stengel 1978, Richter 1978. Stengel et al. 1982 and Richter et al1983). Experimentally, an increase of the viscosity ratio would result in a decrease in the heat transfer (Booker 1976, Booker and Stengel 1978) and hexagonal patterns are solely distinguishable Richter (1978). For the variations of viscosity with exponential or super- exponential temperature, the critical Rayleigh number is nearly constant at small values of the viscosity ratio, then increases at moderate values of the viscosity ratio, increases at moderate values of the viscosity ratio to reach a maximum at a critical viscosity ratio of 2981, and finally decreases in the large viscosity ratio regime above this critical viscosity ratio, the onset of convection is governed by a sub layer that is more unstable than the full layer (see Stengel et al. 1982). Richter et al. (1983) studied the range of viscosity ratio, $10\text{-}10^5$ and showed that the Nusselt number of the variable- viscosity fluid normalized by that for the constant- viscosity fluid of at same Rayleigh number varies by less than 20%. Lam and Bayazitoglu (1987) examined the destabilizing effects of surface tension, linear viscosity variation and internal heat generation and found that viscosity variation plays a more pronounced role in destabilizing the liquid layer.

In this paper, the stationary Rayleigh-Benard instability in a variable viscosity fluid layer with internal heat generation will be studied using linear stability analysis. The boundaries are considered to be insulated to temperature perturbations. A regular perturbation technique with wave number a as a perturbation parameter is used to solve the eigen value problem in a closed form. The influences of temperature-dependent viscosity and internal heating on the stability limit will be analyzed by developing explicit solution.

2. Mathematical Formulation

We consider penetrative convection via internal heating in a system consisting of an infinite horizontal fluid layer of thickness *d* and the z-axis pointing vertically upwards opposing the direction of gravity. The temperatures of the lower and upper boundaries are taken to be uniform and equal to T_l and T_u respectively, with $T_l > T_u$. The governing equations for the fluid layer are:

$$\nabla\cdot\vec{V}=0 \tag{1}$$

$$\rho_0\left(\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}\right)=-\nabla p+\rho_0\vec{g}\left[1-\alpha(T-T_0)\right]+2\nabla.\left[\mu(\nabla.\vec{V}+\nabla.\vec{V}^T)\right] \tag{2}$$

$$\frac{\partial T}{\partial t}+(\vec{V}\cdot\nabla)T=k\nabla^2T+q. \tag{3}$$

where

$$\mu=\mu_0\exp\left[-A(T-T_0)\right] \tag{4}$$

and μ_0 is the dynamic viscosity corresponding to a temperature equal to the mean of temperature at the boundaries. In the above equations, $\vec{V}=(u,v,w)$ is the velocity vector, p is the pressure, T is the temperature, q is the heat source in the fluid layer, κ is the thermal diffusivity, α is the thermal expansion coefficient and ρ_0 is the reference fluid density.

The basic state is quiescent and is of the form

$$u,v,w,p,T \;=\left[0,0,W_0,p_b\;\; z\;\;,T_b\;\; z\;\;\right] \tag{5}$$

The basic steady state is assumed to be quiescent and temperature distributions are found to be

$$T_b\;\; z\;\;=T_0-\left[\left(\frac{T_0-T_u}{d}-\frac{q\,d}{2\kappa}\right)z+\frac{q}{2\kappa}z^2\right] \tag{6}$$

where T_0 is the interface temperature. In order to investigate the stability of the basic solution, infinitesimal disturbances are introduced in the form

$$\vec{V}=\vec{V}',\quad T=T_b+T',\quad p=p_b+p',\quad \rho=\rho_b+\rho',\quad \mu=\mu_b+\mu' \tag{7}$$

where the primed quantities are the perturbations and assumed to be small. Eq.(7)is substituted in Eqs. (1)-(3) and linearized in the usual manner. The pressure term is eliminated from Eq. (2)by taking curl twice on these two equations and only the vertical component is retained. The variables are then nondimensionalized using $d, d^2/\kappa,\ \kappa/d$ and T_0-T_u as the units of length, time, velocity, and temperature in the fluid layer and the non-dimensional disturbance equations are now given by

$$\frac{1}{pr}\frac{\partial}{\partial t}\nabla^2 w=\tilde{f}\,\nabla^4 w+2\frac{\partial\tilde{f}}{\partial z}\nabla^2\frac{\partial w}{\partial z}+\frac{\partial^2\tilde{f}}{\partial z^2}\;\;\nabla^2 w-2\nabla_h^2 w\;\;+R\nabla_h^2 T \tag{8}$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)T=w\;\;1-Ns(1-2z) \tag{9}$$

Where $R=\alpha g\;\;T_0-T_u\;\;d^3/\nu\kappa$ is the Rayleigh number, $Ns=q\,d^2/2\kappa\;\;T_0-T_u$ is the dimensionless heat source strength and $\nabla^2=\nabla_h^2+\partial^2/\partial z^2$ is the Laplacian operator with $\nabla_h^2=\partial^2/\partial x^2+\partial^2/\partial y^2$. The function $\tilde{f}$ representing the temperature dependence of viscosity, is defined as

$$\tilde{f}=\exp\left[B\left(z-\frac{1}{2}\right)\right],\qquad B=\left(\frac{\nu_{\max}}{\nu_{\min}}\right). \tag{10}$$

The boundaries are assumed to be either to be either rigid or free and insulated to temperature perturbation. Thus the appropriate boundary conditions are

For rigid boundary

$$w=Dw=DT=0 \tag{11}$$

for free boundary

$$w=D^2w=DT=0. \tag{12}$$

Since the principle of exchange instabilities holds for thermal convection either in fluid layer or a porous layer. It is reasonable to assume that it holds good even for the present configuration as well. Hence, the time derivatives will be dropped conveniently from Eqs. (8)and(9).Then performing a normal mode expansion of the dependent variables as

$$w,T\;\;=\left[W\;\;z\;\;,\Theta\;\;z\;\;\right]\exp\left[i\;\;lx+my\;\;\right] \tag{13}$$

and substituting them in Eqs. (8) and(9)(with $\partial/\partial t=0$), we obtain the following ordinary differential equations

$$\tilde{f}\;\;D^2-a^2\;\;{}^2W+2D\tilde{f}\;\;D^2-a^2\;\;DW+D^2\tilde{f}\;\;D^2+a^2\;\;W=Ra^2\Theta \tag{14}$$

$$D^2-a^2\;\;\Theta=-W\left[1-Ns\;\;1-2z\;\;\right] \tag{15}$$

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3. Method of Solution

Since the critical wave number is exceedingly small for the assumed temperature boundary conditions (Nield and Bejan 2006) the eigen value problem is solved using a regular perturbation technique with wave number a as a perturbation parameter. Accordingly, the dependent variables are expanded in powers of a^2 in the form

$$W,\Theta \ = \sum_{i=0}^{N} a^2 \ ^{i} \ W_i , \Theta_i \qquad (16)$$

$$\tilde{f}\ D^4 W_0 + 2D\tilde{f}\ D^3 W_0 + D^2\ \tilde{f}\ D^2 W_0 = 0 \qquad (17)$$

$$D^2\theta_0 = -f(z) W_0 \qquad (18)$$

where

$$f(z) = \left[1 - Ns\ \ 1 - 2z\ \ \right] \qquad (19)$$

The boundary conditions are

$$W_0 = D^2 W_0 = D\Theta_0 \ \ = 0 \qquad (20)$$

on the stress-free boundary and

$$W_0 = DW_0 = 0 \qquad (21)$$

on the rigid boundary.

Then solutions to above equations are

$$W_0 = 0 \quad \text{and} \quad \Theta_0 = 1 \qquad (22)$$

First- order equations are

$$D^4 W_1 + 2B\ \ D^3 W_1 + B^2\ \ D^2 W_1 = R\, Exp[-B\ \ z - 1/2\ \ ] \qquad (23)$$

$$D^2\Theta_1 = 1 - f(z) W_1. \qquad (24)$$

The general solution of (23) is

$$W_1 = R\left[\ C_1 + C_2 z + C_3 e^{-Bz} + C_4\ z e^{-Bz} + \frac{z^2}{2B^2} e^{-B\ \ z-1/2}\ \ \right] \qquad (25)$$

where C_1, C_2, C_3 and C_4 are constants and they have to determined using the appropriate boundary conditions.

Case (i) Both Boundaries are Rigid

The boundary conditions are;

$$W_1 = DW_1 \ = 0 \qquad at \quad z = 0,1 \qquad (26)$$

The differential Equation (24) involving $D^2\Theta_1$ provide the solvability requirement which is given by

$$\int_0^1 f(z) W_1\, dz = 1 \qquad (27)$$

The expressions for W_1 is back substituted into Eq. (27) and integrated to yield an expression for the critical Rayleigh number R_c , which is given by

$$R_c = \frac{1}{C_1 + C_2\left(\frac{1}{2}+\frac{Ns}{6}\right) + C_3\ e^{B}\ 1-Ns\ -1\ + C_4\ e^{B}-1-Ns\ 4e^{B}-1\ + d_1} \qquad (28)$$

where

$$C_1 = -\frac{e^{B/2}}{2B^2}\left(\frac{e^{B}-B-1}{1-2e^{B}+e^{2B}-B^2e^{B}}\right),\quad C_2 = \frac{e^{B/2}}{2B^2}\left(\frac{2-2e^{B}+B+Be^{B}}{e^{2B}+2e^{B}-B^2e^{B}-1}\right)$$

$$C_3 = \frac{e^{B/2}}{2c^2}\left(\frac{e^{B}-B-1}{1+e^{2B}-2e^{B}-B^2e^{B}}\right),\ C_4 = \frac{e^{B/2}}{2B^2}\left(\frac{2-2e^{B}+2B+B^2}{2e^{B}-e^{2B}+B^2e^{B}-1}\right)$$

$$d_1 = \frac{1}{2B^3}\left[\ 2B-2\ e^{B/2}+2e^{-B/2}+Ns\ \ 2-2B\ e^{-B/2}-2e^{B/2}\ \right].$$

From Eqs. (28), we note that in the absence of internal heating (i.e., $Ns \to 0$) and constant viscosity (i.e., $B \to 0$), $R_c \to 720$ which is the known exact value (Nield 1987]).

Case (ii) Both Boundaries are Free

The boundary condition are;

$$W_1 = D^2W_1 = 0 \quad at \quad z = 0,1 \qquad (29)$$

The expression for the critical Rayleigh number R_c , which is given by

$$R_c = \frac{1}{C_1 + C_2\left(\frac{1}{2}+\frac{Ns}{6}\right) + C_3\ e^{B}\ 1-Ns\ -1\ + C_4\ e^{B}-1-Ns\ 4e^{B}-1\ + d_1} \qquad (30)$$

where

$$C_1 = -\frac{e^{B/2}}{2B^2}\left(\frac{e^{B}-B-1}{1-2e^{B}+e^{2B}-B^2e^{B}}\right),\quad C_2 = \frac{e^{B/2}}{2B^2}\left(\frac{2-2e^{B}+B+Be^{B}}{e^{2B}+2e^{B}-B^2e^{B}-1}\right)$$

$$C_3 = \frac{e^{B/2}}{2B^2}\left(\frac{e^{B}-B-1}{1+e^{2B}-2e^{B}-B^2e^{B}}\right),\quad C_4 = \frac{e^{B/2}}{2c^2}\left(\frac{2-2e^{B}+2B+B^2}{2e^{B}-e^{2B}+B^2e^{B}-1}\right).$$

From Eqs. (30), we note that in the absence of internal heating (i.e., $Ns \to 0$) and constant viscosity (i.e., $B \to 0$), $R_c \to 120$ which is the known exact value (Nield 1987).

Case (iii) Lower Boundary is Rigid and Upper Boundary is Free

The boundary condition are;

$$W_1 = DW_1 = 0 \quad at \quad z = 0 \qquad (31)$$

$$W_1 = D^2W_1 = 0 \quad at \quad z = 1. \qquad (32)$$

The expression for the critical Rayleigh number R_c , which is given by

$$R_c = \frac{1}{C_1 + C_2\left(\frac{1}{2}+\frac{Ns}{6}\right) + C_3\left(e^{B}(1-Ns)-1\right) + C_4\left(e^{B}-1-Ns(4e^{B}-1)\right) + d_1} \qquad (33)$$

where

$$C_1 = -\frac{e^{B/2}}{2B^2}\left(\frac{e^B - B - 1}{1-2e^B+e^{2B}-B^2e^B}\right),\quad C_2 = \frac{e^{c/2}}{2c^2}\left(\frac{2-2e^c+c+ce^c}{e^{2c}+2e^c-c^2e^c-1}\right)$$

$$C_3 = \frac{e^{B/2}}{2B^2}\left(\frac{e^B - B - 1}{1+e^{2B}-2e^B-B^2e^B}\right),\quad C_4 = \frac{e^{B/2}}{2B^2}\left(\frac{2-2e^B+2B+B^2}{2e^B-e^{2B}+B^2e^B-1}\right).$$

From Eqs. (33), we note that in the absence of internal heating (i.e., $Ns \to 0$) and constant viscosity (i.e., $B \to 0$), $R_c \to 320$ which is the known exact value (Nield 1987).

4. Results and Discussion

The effect of internal heat generation on the criterion for the onset of Rayleigh-Benard instability in a variable viscosity fluid layer is investigated theoretically. The resulting eigen value problem is solved using a regular perturbation technique with wave number *a* as a perturbation parameter. The following three different types of boundary conditions are considered for discussion namely,

Case (i): Both boundaries are rigid
Case (ii): Both boundaries are free
Case (iii): Lower boundary is rigid and upper boundary is free.

Figure 1 shows the variation of critical Rayleigh number R_c with the viscosity parameter B for different values of Ns for the boundaries of case(i), it is observed that in the absence of internal heating($Ns = 0$) the critical Rayleigh number R_c increases initially, with $B,$ reaches maximum and then decreases with further increase in the value of B and thus three regions are distinguished as observed in the case of isothermal boundary (see Stengel et al. 1982) R_c increases only negligibly with B for small values of $B;$ increases significantly for B up to about 8 or 9, at which maximum values R_c are reached; rapidly decreasing trends are found for values of B above 9. Figure 2 depicts the perturbed vertical velocity eigen functions W for different values of the viscosity parameter B for $Ns = 0$. It is noted that the appearance of newly formed sub layer, which first occurs at the maximum critical Rayleigh number R_c with associated viscosity parameter, continues to manifest itself after then, becoming dominant at the critical state. As B is further increased, the viscously suppressive effects of main fluid layer above shorten the depth of sub layer and R_c then decreases with $B.$ For the larger values of viscosity parameter B the vertical velocity vanishes at the lower part of fluid layer. Further, in the Fig.4.1 with an increase of internal heating in the fluid layer the critical Rayleigh number R_c decreases with increases the value of $B.$ Figure 4.3 depicts the perturbed vertical velocity eigen functions W for different values of Ns for $B = 3$. It is noted that the convection is closer to upper boundary with an increase of internal heating, thereby the effect internal heating in the fluid layer is todestabilize the system.

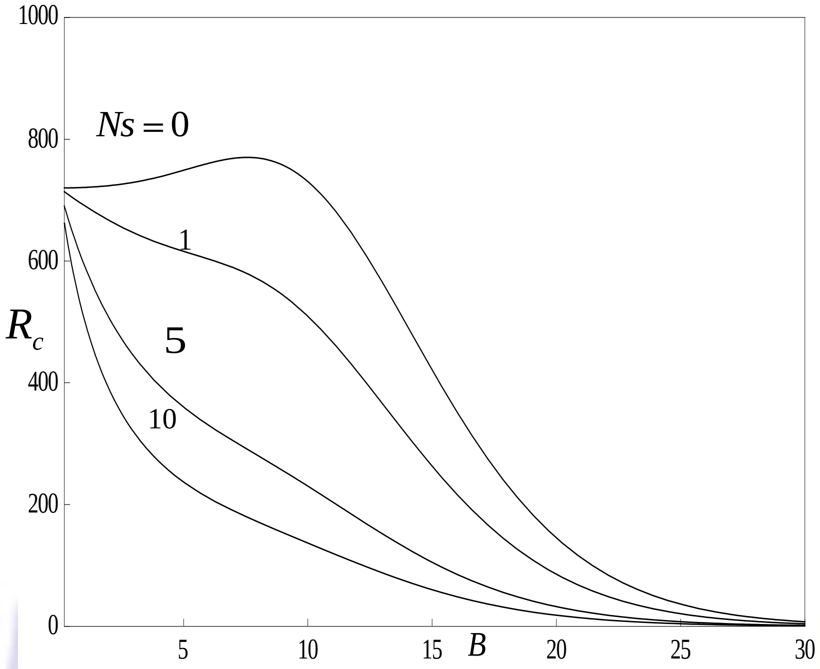

Fig.1 Critical Rayleigh number versus B for different values of Ns for rigid-rigid boundaries.

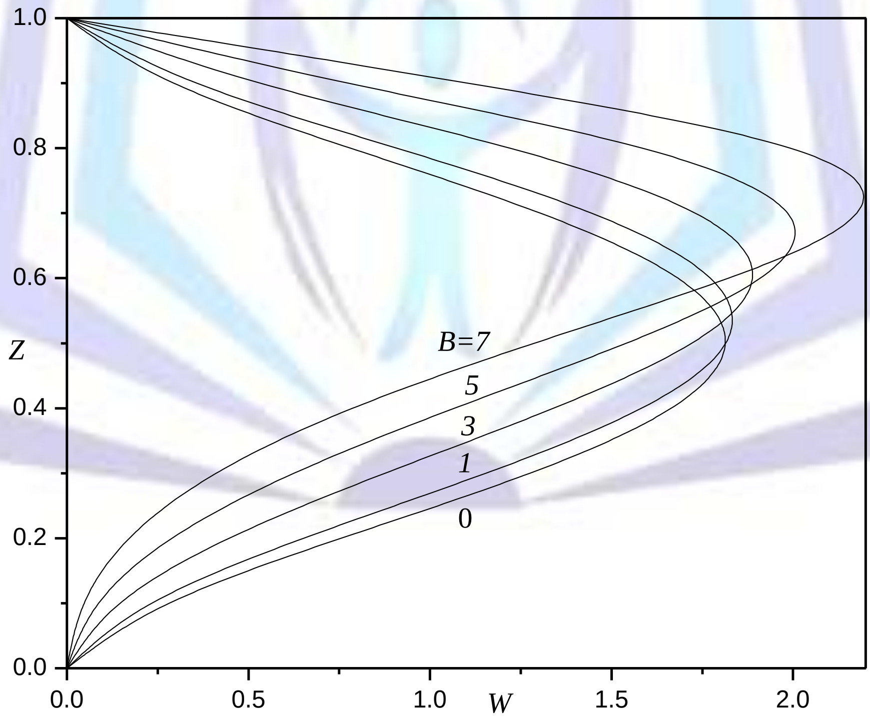

Fig.2 Perturbed velocity eigen functions W for different values of B with $Ns = 0$ for rigid-rigid boundaries.

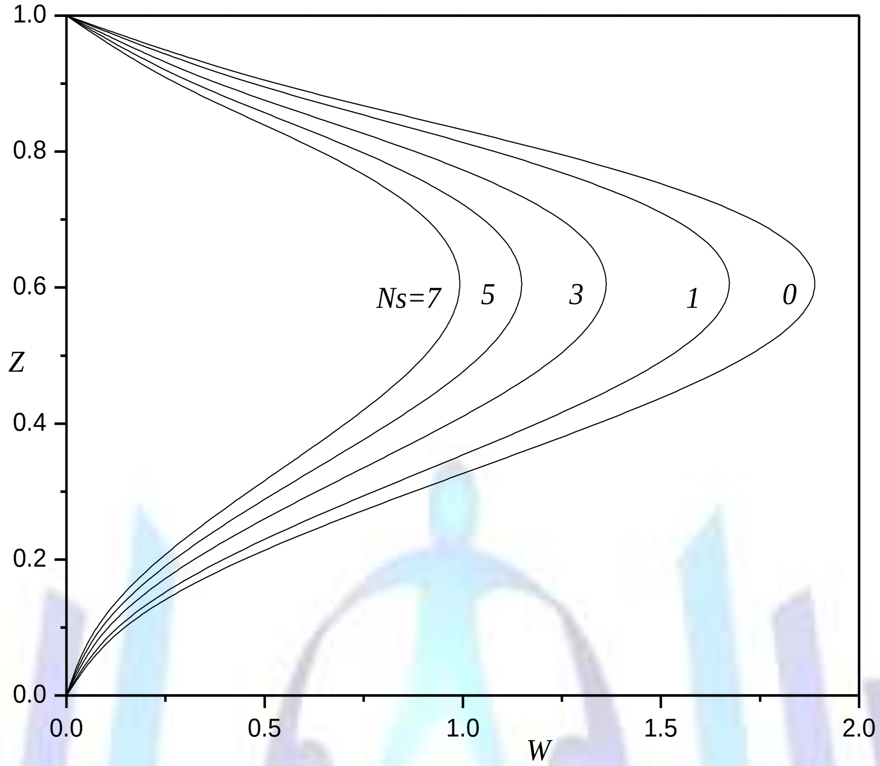

Fig.3 Perturbed velocity eigen functions W for different values of Ns with $B=3$ for rigid-rigid boundaries.

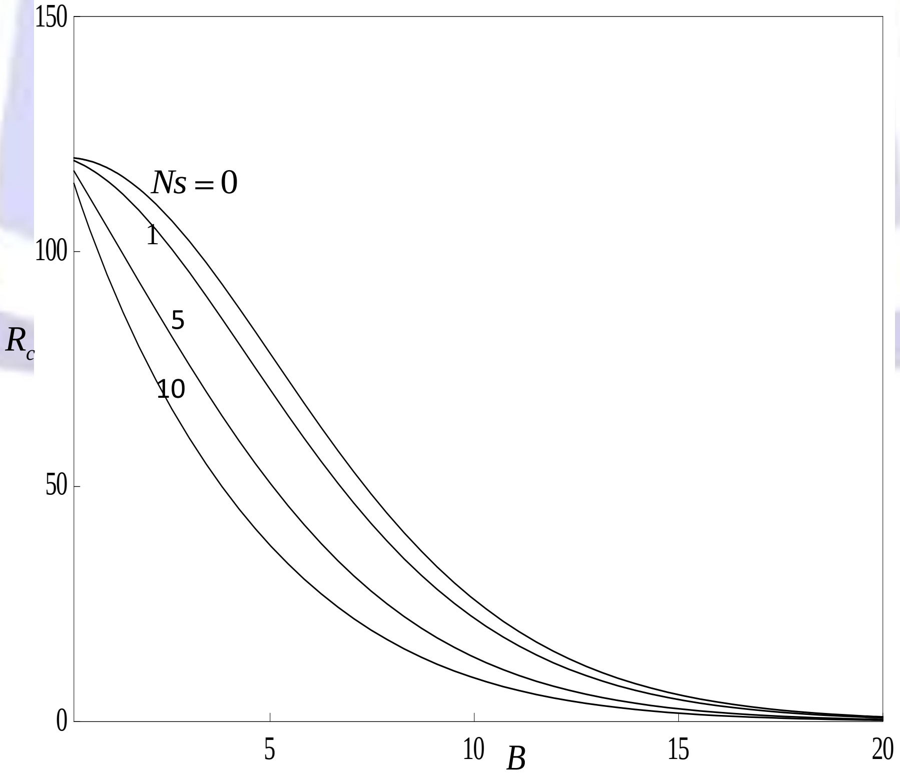

Fig. 4 Critical Rayleigh number versus B for different values of Ns for free-free boundaries.

Figure 4 shows the variation of critical Rayleigh number R_c with the viscosity parameter B for different values of Ns for the boundaries of case(ii), it is observed that in the absence and presence of internal heating the critical Rayleigh number R_c shown to decrease monotonically with $B,$ showing important thermal effects from both above and below and also the comparatively negligible viscous effects in the lower part of the fluid layer. As shown in the Fig.5, the onset of convection is governed by the full layer rather than sub layer. Figure 6 depicts the perturbed vertical velocity eigen functions W for different values of Ns for $B=3.$ It is noted that the convection is closer to upper boundary with an increase of internal heating.

Figure 7shows the variation of critical Rayleigh number R_c with the viscosity parameter B for different values of Ns for the boundaries of case(iii),It is observed that in the absence of internal heating($Ns=0$) and small value of $Ns=1$, the critical Rayleigh number R_c increases initially, then decreases rapidly. The higher values of Ns, R_c decreases monotonically as the value of B increases. Figure 8 depicts the perturbed vertical velocity eigen functions W for different values of the viscosity parameter B for $Ns=0$. It is noted that the critical state does not vanish throughout the entire fluid layer, except at its boundaries. Further, convection is closer to upper boundary with an increase of internal heating (Fig. 9).

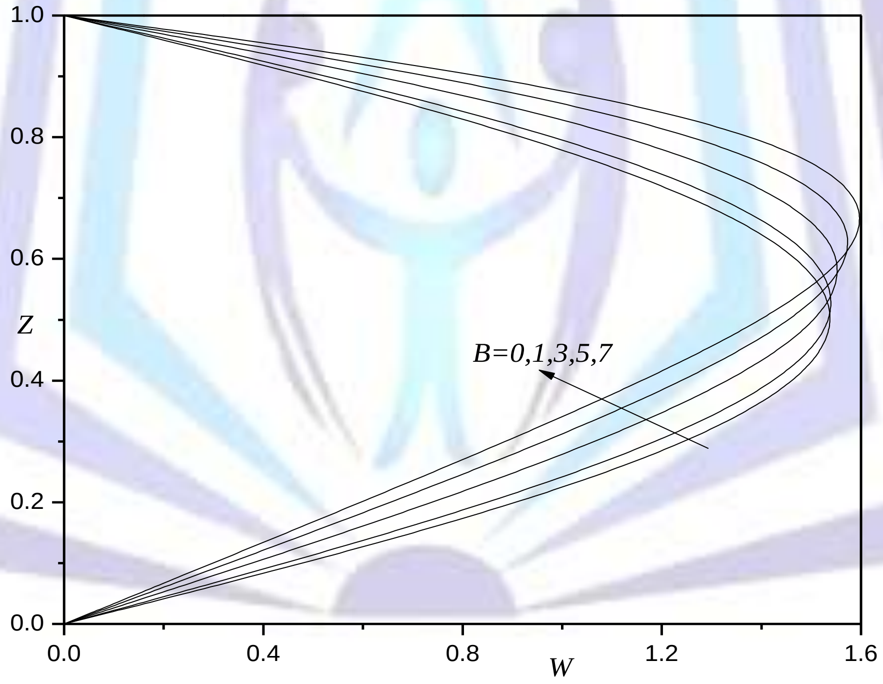

Fig.5 Perturbed velocity eigen functions W for different values of B with $Ns=0$ for free-free boundaries.

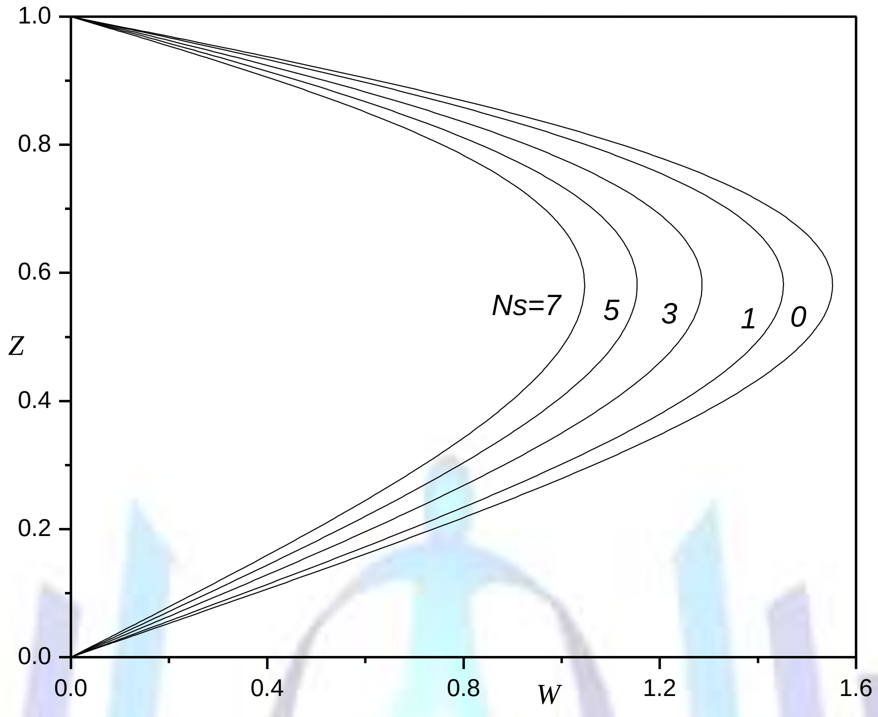

Fig. 6 Perturbed velocity eigen functions W for different values of Ns with $B=3$ for free-free boundaries.

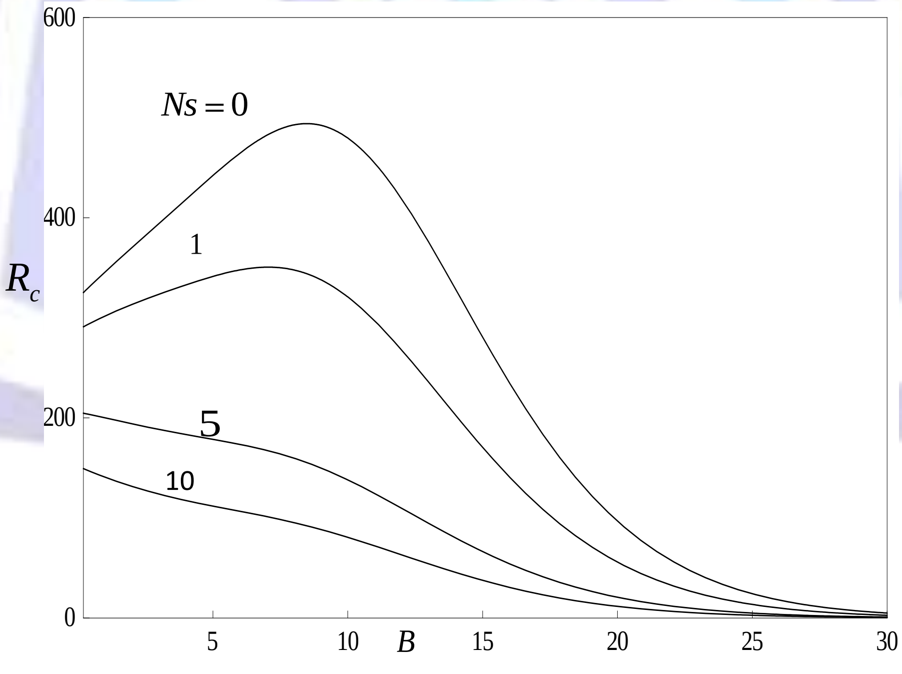

Fig. 7 Critical Rayleigh number versus B for different values of Ns for rigid-free boundaries

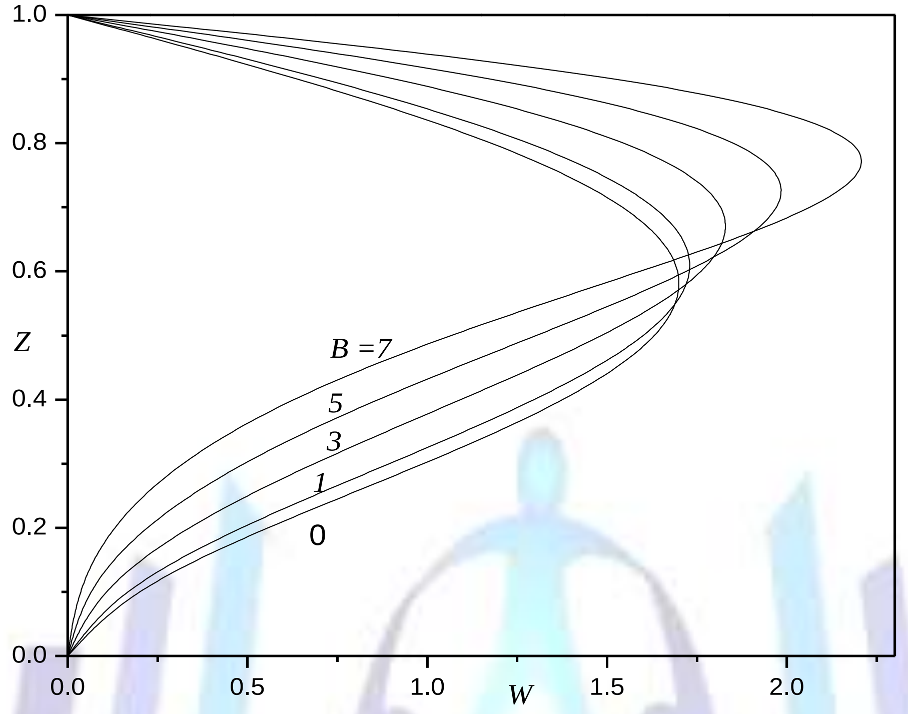

Fig.8 Perturbed velocity eigen functions W for different values of B with $Ns = 0$ for rigid-free boundaries.

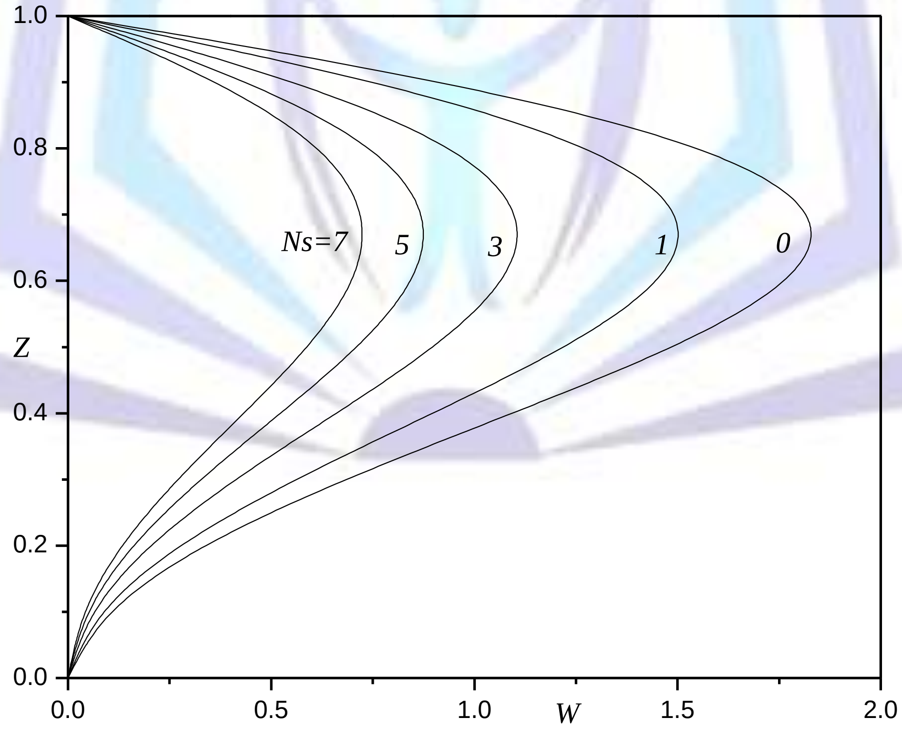

Fig.9 Perturbed velocity eigen functions W for different values of Ns with $B = 3$ for rigid-free boundaries.

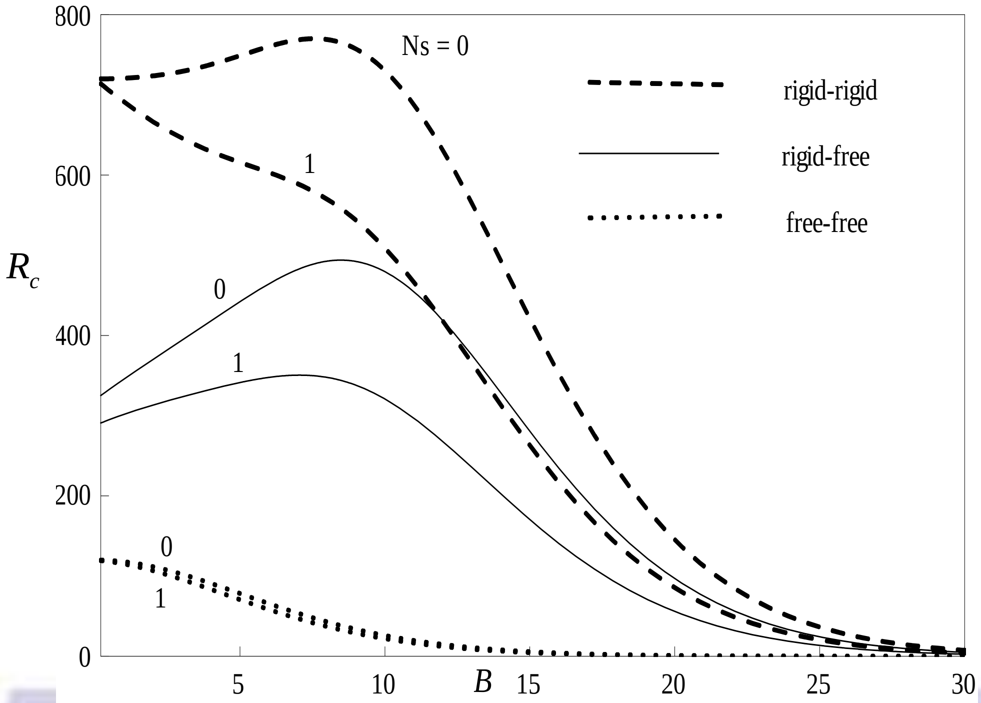

Fig. 10 Critical Rayleigh number versus B for different values of Ns for all three boundaries.

Figure 10 shows the variation of critical Rayleigh number R_c with the viscosity parameter B for different values of Ns for all three different types of boundary conditions. The curves of R_c for different Ns for case (i) lie above case (iii), while those of case (ii) lie below case (iii) boundaries. Hence the case (i) boundary combination is the most stable configuration, compared to case(ii) and case(iii) boundaries in the both absence and presence of internal heating the fluid layer.

5. Conclusions

The onset of penetrative convection via internal heating in a fluid layer is studied with an exponential viscosity variation. The resulting eigen value problem is solved using a regular perturbation technique with wave number *a* as a perturbation parameter. The effect of internal heating in a fluid layer is to hasten the onset of convection irrespective of boundaries considered. For the boundaries of case(i), the critical Rayleigh number R_c increases initially, with $B,$ reaches maximum and then decreases with further increases the value of B in the absence of internal heating($Ns = 0$). While in the presence of internal heating R_c decreases with increases of the value $B.$ For the boundaries of case(ii), the critical Rayleigh number R_c decrease monotonically with $B,$ in both absence and presence of internal heating in the fluid layer. For the boundaries of case(iii), in the absence of internal heating($Ns = 0$) and small value of $Ns = 1,$ the critical Rayleigh number R_c increases initially, then decreases rapidly. At the higher values of Ns, R_c decreases monotonically as the value of B increases. The case (i) boundary combination is the most stable configuration, compared to case(ii) and case(iii) boundaries in the absence and presence of internal heating in the fluid layer. Thus it is possible to either augment or suppress the onset of Rayleigh-Benard convection by suitably choosing the parametric values.

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