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On Some Double Integrals Involving $\overline{H}$-Function of Two Variables and Spheroidal Functions

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Abstract

The present paper evaluates certain double integrals involving $\overline{H}$-function of two variables [21] and Spherodial functions [23]. These double integrals are of most general character known so far and can be suitably specialized to yield a number of known or new integral formulae of much interest to mathematical analysis which are likely to prove quite useful to solve some typical boundary value problems.

Key words: $\overline{H}$-function; $\overline{H}$-function of two variables; Spheroidal function.

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1. Introduction

The $\overline{H}$ -function occurring in the paper is defined and represented by Buschman and Srivastava [3] as follows :

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \qquad (1.1)$$

where $$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (1.2)$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,...,p)$ and $b_j(j=1,...,Q)$ are complex parameters, $\alpha_j \ge 0(j=1,...,P), \beta_j \ge 0(j=1,...,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,...,N)$ and $B_j(j=N+1,...,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$ -function given by equation (1.1) have been given by (Buschman and Srivastava).

$$\Omega \equiv \sum_{j=1}^{M}\left|\beta_j\right| + \sum_{j=1}^{N}\left|A_j\alpha_j\right| - \sum_{j=M+1}^{Q}\left|\beta_jB_j\right| - \sum_{j=N+1}^{P}\left|\alpha_j\right| > 0 \qquad (1.3)$$

and $$\left|\arg(z)\right| < \frac{1}{2}\pi\Omega \qquad (1.4)$$

The behavior of the $\overline{H}$ -function for small values of $|z|$ follows easily from a result recently given by (Rathie [18],p.306,eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}[z] = 0\left(|z|^{\gamma}\right), \gamma = \min_{1\le j\le N}\left[\operatorname{Re}\left(b_j / \beta_j\right)\right], |z| \to 0 \qquad (1.5)$$

If we take $A_j = 1(j=1,2,...,N), B_j = 1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}[.]$ reduces to the Fox's H -function [].

The following series representation for the $\overline{H}$ -function will be required in the sequel [see Rathie,[18] pp.305-306,eq.(6.8)]:

$$\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] =$$

$$\frac{\sum_{h=1}^{M}\sum_{r=0}^{\infty}\prod_{\substack{j=1\\j\ne h}}^{M}\Gamma\left(b_j-\beta_j\xi_{h,r}\right)\prod_{j=1}^{N}\left\{\Gamma\left(1-a_j+\alpha_j\xi_{h,r}\right)\right\}^{A_j}(-1)^r z^{\xi_{h,r}}}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b_j+\beta_j\xi_{h,r}\right)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma\left(a_j-\alpha_j\xi_{h,r}\right)r!\beta_h} \qquad (1.6)$$

Where

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$$\xi_{h,r} = \frac{(b_h + r)}{\beta_h}.$$

The $\overline{H}$ -function of two variables

The $\overline{H}$ -function of two variables introduced by Singh and Mandia [21] will be defined and represented in the following manner:

$$\overline{H}[x,y] = \overline{H}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2} \phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1.7)$$

Where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)} \qquad (1.8)$$

$$\phi_2(\xi) = \frac{\prod_{j=1}^{n_2}\{\Gamma(1-c_j+\gamma_j\xi)\}^{K_j}\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=m_2+1}^{q_2}\{\Gamma(1-d_j+\delta_j\xi)\}^{L_j}} \qquad (1.9)$$

$$\phi_3(\eta) = \frac{\prod_{j=1}^{n_3}\{\Gamma(1-e_j+E_j\eta)\}^{R_j}\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=m_3+1}^{q_3}\{\Gamma(1-f_j+F_j\eta)\}^{S_j}} \qquad (1.10)$$

Where x and y are not equal to zero (real or complex), and an empty product is interpreted as unity p_i, q_i, n_i, m_j are non-negative integers such that $0 \le n_i \le p_i, o \le m_j \le q_j (i=1,2,3; j=2,3)$. All the $a_j(j=1,2,...,p_1), b_j(j=1,2,...,q_1), c_j(j=1,2,...,p_2), d_j(j=1,2,...,q_2),$

$e_j(j=1,2,...,p_3), f_j(j=1,2,...,q_3)$ are complex parameters. $\gamma_j \ge 0(j=1,2,...,p_2), \delta_j \ge 0(j=1,2,...,q_2)$ (not all zero simultaneously), similarly $E_j \ge 0(j=1,2,...,p_3), F_j \ge 0(j=1,2,...,q_3)$ (not all zero simultaneously). The exponents $K_j(j=1,2,...,n_3), L_j(j=m_2+1,...,q_2), R_j(j=1,2,...,n_3), S_j(j=m_3+1,...,q_3)$ can take on non-negative values.

The contour L_1 is in ξ -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma(d_j-\delta_j\xi)(j=1,2,...,m_2)$ lie to the right and the poles of $\Gamma\{(1-c_j+\gamma_j\xi)\}^{K_j}(j=1,2,...,n_2), \Gamma(1-a_j+\alpha_j\xi+A_j\eta)(j=1,2,...,n_1)$ to the left of the contour. For $K_j(j=1,2,...,n_2)$ not an integer, the poles of gamma functions of the numerator in (1.9) are converted to the branch points.

The contour L_2 is in η -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma(f_j-F_j\eta)(j=1,2,...,m_3)$ lie to the right and the poles of $\Gamma\{(1-e_j+E_j\eta)\}^{R_j}(j=1,2,...,n_3), \Gamma(1-a_j+\alpha_j\xi+A_j\eta)(j=1,2,...,n_1)$ to the left of the contour. For $R_j(j=1,2,...,n_3)$ not an integer, the poles of gamma functions of the numerator in (1.10) are converted to the branch points.

The functions defined in (1.7) is an analytic function of x and y, if

$$U=\sum_{j=1}^{p_1}\alpha_j+\sum_{j=1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j-\sum_{j=1}^{q_2}\delta_j<0 \tag{1.11}$$

$$V=\sum_{j=1}^{p_1}A_j+\sum_{j=1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j-\sum_{j=1}^{q_3}F_j<0 \tag{1.12}$$

The integral in (1.7) converges under the following set of conditions:

$$\Omega=\sum_{j=1}^{n_1}\alpha_j-\sum_{j=n_1+1}^{p_1}\alpha_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_jL_j+\sum_{j=1}^{n_2}\gamma_jK_j-\sum_{j=n_2+1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j>0 \tag{1.13}$$

$$\Lambda=\sum_{j=1}^{n_1}A_j-\sum_{j=n_1+1}^{p_1}A_j+\sum_{j=1}^{m_2}F_j-\sum_{j=m_2+1}^{q_2}F_jS_j+\sum_{j=1}^{n_3}E_jR_j-\sum_{j=n_2+1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j>0 \tag{1.14}$$

$$|\arg x|<\frac{1}{2}\Omega\pi,\ |\arg y|<\frac{1}{2}\Lambda\pi \tag{1.15}$$

The behavior of the $\overline{H}$ -function of two variables for small values of $|z|$ follows as:

$$\overline{H}[x,y]=0(|x|^{\alpha}|y|^{\beta}),\max\{|x|,|y|\}\to 0 \tag{1.16}$$

Where

$$\alpha=\min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{d_j}{\delta_j}\right)\right]\qquad \beta=\min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{f_j}{F_j}\right)\right] \tag{1.17}$$

For large value of $|z|$,

$$\overline{H}[x,y]=0\{|x|^{\alpha'},|y|^{\beta'}\},\min\{|x|,|y|\}\to 0 \tag{1.18}$$

Where

$$\alpha'=\max_{1\le j\le n_2}\operatorname{Re}\left(K_j\frac{c_j-1}{\gamma_j}\right),\ \beta'=\max_{1\le j\le n_3}\operatorname{Re}\left(R_j\frac{e_j-1}{E_j}\right) \tag{1.19}$$

Provided that $U<0$ and $V<0$.

If we take $K_j=1(j=1,2,...,n_2),L_j=1(j=m_2+1,...,q_2),R_j=1(j=1,2,...,n_3),S_j=1(j=m_3+1,...,q_3)$ in (1.7), the $\overline{H}$ -function of two variables reduces to H -function of two variables due to [15].

Spheroidal function:

Spheroidal function $\psi_{\alpha n}(c,z)$ of general order $\alpha>-1$ is defined and investigated by Stratton [] and later by Chu and Stratton [4] are those solutions of the differential equation:

$$\left(1-z^2\right)\psi''_{\alpha n}(c,z)-2(\alpha+1)z\psi'_{\alpha n}(c,z)+\left(b_{\alpha n}-c^2z^2\right)\psi_{\alpha n}(c,z)=0 \tag{1.20}$$

That remains finite at the singular points $z=\pm 1$.

The spheroidal function can be expanded in Bessel function on $(-\infty,\infty)$ ([19], p. 190)

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$$\psi_{\alpha n}(c,z)=\frac{i^n\sqrt{2\pi}}{V_{\alpha n}(c)}\sum_{k=0,1}^{\infty}{}' \frac{a_k\left(c/\alpha n\right)J_{k+\alpha+\frac{1}{2}}(cz)}{(cz)^{\alpha+\frac{1}{2}}} \tag{1.21}$$

With eight eigen –values $V_{\alpha n}(c)$, valid for $\alpha>-1$, whereas the prime over the summation sigh indicates that the summation is taken over only even or odd values of k as n is even or odd.

A recursion relationship ([23], (9)) for determining the a_k, s in (1.21) and the eigen-values $b_{\alpha n}(c)$ are obtained from the differential equation (1.20).

If (cz) is real and finite $\frac{J_{k+\alpha+\frac{1}{2}}(cz)}{(cz)^{\alpha+\frac{1}{2}}}$ is bounded, hence it follows by M-test that the series in (1.21) is absolutely and uniformly convergent. Moreover it represents a continuous function for all cz.

If $c\to 0$ and $z\to\infty$ such that cz remains finite and the normalization is chosen to be such that

$$\frac{\sqrt{2\pi}}{V_{\alpha n}}\sum_{k=0,1}^{\infty_1}{}' i^n a_k\left(c/\alpha n\right)=1, \tag{1.22}$$

The function $\psi_{\alpha n}(c,z)$ reduces to $\frac{J_{\alpha+n+\frac{1}{2}}(cz)}{(cz)^{\alpha+\frac{1}{2}}}$ since $a_k\left(c/\alpha n\right)\to 0, k\neq n$.

The following results ([5], p. 172; [9], p. 145; [13], p. 226) in the sequel will be used during the proof of our main results in a little simplification:

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}f\left(Ax^\gamma+By^\delta\right)dxdy=\frac{A^{-\frac{\alpha}{\gamma}}B^{-\frac{\beta}{\delta}}}{\gamma\delta}\frac{\Gamma\left(\frac{\alpha}{\gamma}\right)\Gamma\left(\frac{\beta}{\delta}\right)}{\Gamma\left(\frac{\alpha+\beta}{\gamma+\delta}\right)}\int_0^\infty z^{\left(\frac{\alpha}{\gamma}+\frac{\beta}{\delta}-1\right)}f(z)dz \tag{1.23}$$

$\min\{A,B,\gamma,\delta\},\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$.

$$\int_0^\infty x^{\rho-1}\overline{H}_{p,q}^{m,0}\left[cx\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\frac{c^{-\rho}\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j\rho\right)}{\prod_{j=m+1}^{q}\left\{\Gamma\left(1-b_j-\beta_j\rho\right)\right\}^{B_j}\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j\rho\right)} \tag{1.24}$$

Provided $\mathrm{Re}\left[\rho+B_j\left(\frac{b_j}{\beta_j}\right)\right]>0,\ j=1,2,...,m;|\arg c|<\frac{1}{2}\Omega\pi$, where

$\Omega=\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q}B_j\beta_j-\sum_{j=1}^{p}\alpha_j>0$.

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$$2^{\mu}\overline{J}_{v}(z)=z^{\mu}\overline{H}_{0,2}^{1,0}\left[\frac{z^2}{2}\left|\begin{matrix}-\\ \left(\frac{\mu+v}{2},1\right),\left(\frac{\mu+v}{2},1;1\right)\end{matrix}\right.\right] \quad (1.25)$$

$$\overline{H}_{P,Q}^{M,N}\left[x^{c}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{c}\overline{H}_{P,Q}^{M,N}\left[x\left|\begin{matrix}(a_j,\alpha_j/c;A_j)_{1,N},(a_j,\alpha_j/c)_{N+1,P}\\ (b_j,\beta_j/c)_{1,M},(b_j,\beta_j/c;B_j)_{M+1,Q}\end{matrix}\right.\right];c>0 \quad (1.26)$$

2. Main Results

We establish the following double integrals:

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\overline{H}_1\left[ux^s y^k\left(Ax^{\gamma}+By^{\delta}\right)^{\sigma_1},v\left(Ax^{\gamma}+By^{\delta}\right)^{\sigma_2}\right]f\left(Ax^{\gamma}+By^{\delta}\right)dxdy$$

$$=\frac{A^{-\frac{\alpha}{\gamma}}B^{-\frac{\beta}{\delta}}}{\gamma\delta}\int_0^\infty z^{\frac{\alpha}{\gamma}+\frac{\beta}{\delta}-1}f(z)\overline{H}_{p_1,q_1:p_2+2,q_2+1;p_2,q_2}^{0,0:\ m_2,n_2+2:\ m_3,n_2}$$

$$\left[\begin{matrix}Mz^N\\ vz^{\sigma_2}\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},\left(1-\frac{\alpha}{\gamma},\frac{s}{\gamma};1\right),\left(1-\frac{\beta}{\delta},\frac{k}{\delta};1\right),(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{\alpha}{\gamma}-\frac{\beta}{\delta},\frac{s}{\gamma}+\frac{k}{\delta};1\right),(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]dz \quad (2.1)$$

Where, for convenience, $M=uA^{-\frac{s}{\gamma}}B^{-\frac{k}{\delta}},N=\frac{s}{\gamma}+\frac{k}{\delta}+\sigma_1$ and the function f is so prescribed that the integral (2.1) converges.

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\left(Ax^{\gamma}+By^{\delta}\right)^{\sigma_3}\overline{H}_{p,q}^{m,0}\left[\left(Ax^{\gamma}+By^{\delta}\right)^{\sigma_4}\right]$$

$$\overline{H}_1\left[ux^s y^k\left(Ax^{\gamma}+By^{\delta}\right)^{\sigma_1},v\left(Ax^{\gamma}+By^{\delta}\right)^{\sigma_2}\right]dxdy$$

$$=\frac{A^{-\frac{\alpha}{\gamma}}B^{-\frac{\beta}{\delta}}W^{-\theta}}{\gamma\delta\sigma_4}\overline{H}_{p_1+q,q_1+p:p_2+2,q_2+2:p_3,q_3}^{0,m:m_2,n_2+2:m_3,n_3}\left[\begin{matrix}MW^{-\phi}\\ vW^{-\frac{\sigma_2}{\sigma_4}}\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},\left(1-b_q-\beta_q\theta;\beta_q\phi,\beta_q\frac{\sigma_2}{\sigma_4}\right),(c_j,\gamma_j;K_j)_{1,n_2},\left(1-\frac{\alpha}{\gamma},\frac{s}{\gamma};1\right),\left(1-\frac{\beta}{\delta},\frac{k}{\delta};1\right),(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},\left(1-a_p-\alpha_p\theta;\alpha_p\phi,\alpha_p\frac{\sigma_2}{\sigma_4}\right)(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{\alpha}{\gamma}-\frac{\beta}{\delta},\frac{s}{\gamma}+\frac{k}{\delta};1\right),(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right] \quad (2.2)$$

Where for convenience, $\theta=\left(\frac{\alpha}{\gamma}+\frac{\beta}{\delta}+\sigma_3\right)\Big/\sigma_4$, $\phi=\left(\frac{s}{\gamma}+\frac{k}{\delta}+\sigma_1\right)\Big/\sigma_4$, $M=uA^{-\frac{s}{\gamma}}B^{-\frac{k}{\delta}}$

provided that $|\arg W|<\frac{1}{2}\Omega\pi$ where

$$\Omega=\sum_{j=1}^{m}\beta_j+\sum_{j=m+1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j>0,$$

$$\mathrm{Re}\left[\frac{\alpha}{\gamma}+\frac{\beta}{\delta}+\sigma_3+\sigma_4\frac{b_j}{\beta_j}+\sigma_1 L_j\frac{d_i}{D_i}+\sigma_2 S_{j'}\frac{f_{j'}}{F_{j'}}\right]>0$$

$\left(j=1,\ldots,m; i=1,\ldots,m_2; j'=1,\ldots,m_3\right); \sigma_1,\sigma_2,\sigma_3,\sigma_4$ are all positive quantities and the conditions (1.11) to (1.15), with x replaced by M and y replaced by v, are satisfied.

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\overline{H}_1\left[ux^s y^k\left(Ax^\gamma+By^\delta\right)^{\sigma_1}, v\left(Ax^\gamma+By^\delta\right)^{\sigma_2}\right]\left(Ax^\gamma+By^\delta\right)^{\sigma_3}$$

$$\psi_{\mu n}\left[c^{\sigma_4/2}, 2W\left(\frac{Ax^\gamma+By^\delta}{c}\right)^{\sigma_4/2}\right]dxdy = \frac{A^{-\frac{\alpha}{\gamma}}B^{-\frac{\beta}{\delta}}i^n\sqrt{2\pi}\,W^{-2\theta}}{2^{\mu+\frac{1}{2}}\gamma\delta\sigma_4 V_{\alpha n}\left(c^{\sigma_4/2}\right)}$$

$$\sum_{r=0,1}^{\infty}{}' a_\gamma\left(\frac{c^{\sigma_4/2}}{\mu n}\right)\overline{H}^{0,1:m_2,n_2+2:m_3,n_3}_{p_1+2,q_1:p_2+2,q_1+1:p_3,q_3}\left[\begin{matrix} MW^{-2\phi} \\ vW^{-\frac{2\sigma_2}{\sigma_4}}\end{matrix}\right.$$

$$\left.\begin{matrix}\left(a_{p_1},\alpha_{p_1};A_{p_1}\right),\left(1\pm\frac{1}{2}[1+r+2\mu-2\theta];\phi,\frac{\sigma_2}{\sigma_4}\right),\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(1-\frac{\alpha}{\gamma},\frac{s}{\gamma};1\right),\left(1-\frac{\beta}{\delta},\frac{k}{\delta};1\right),\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3} \\ \left(b_{q_1},\beta_{q_1};B_{q_1}\right),\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(1-\frac{\alpha}{\gamma}-\frac{\beta}{\delta},\frac{s}{\gamma}+\frac{k}{\delta};1\right),\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{matrix}\right] \quad (2.3)$$

Where for convenience, $\theta = \left(\frac{\alpha}{\gamma}+\frac{\beta}{\delta}+\sigma_3\right)\Big/\sigma_4$, $\phi = \left(\frac{s}{\gamma}+\frac{k}{\delta}+\sigma_1\right)\Big/\sigma_4$, $M = uA^{-\frac{s}{\gamma}}B^{-\frac{k}{\delta}}$,

provided that $\mathrm{Re}\left[\frac{\alpha}{\gamma}+\frac{\beta}{\delta}+\sigma_3+\sigma_1 L_j\frac{d_i}{D_i}+\sigma_2 S_{j'}\frac{f_{j'}}{F_{j'}}+r\frac{\sigma_4}{2}\right]>0$,

$\left(i=1,\ldots,m_2; j'=1,\ldots,m_3; r=0,1,\ldots\right); c,\sigma_1,\sigma_2,\sigma_3,\sigma_4$ are all positive quantities; W is not equal to zero; and the conditions (1.11) to (1.15) with x replaced by M and y replaced by v are satisfied.

Proof of (2.1): To prove the integral relation (2.1), we first replace $\overline{H}_1$ by its Mellin-Barnes double contour integral from (1.7) with $m_1 = 0$. On inverting the order of integration, which is justified due to absolute convergence of the integrals involved in the process, we obtain

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)U_1(\xi)U_2(\eta)u^\xi v^\eta\left[\int_0^\infty\int_0^\infty x^{\alpha+\xi s-1}y^{\beta+k\xi-1}\right.$$

$$\left.\left(Ax^\gamma+By^\delta\right)^{\sigma_1\xi+\sigma_2\eta} f\left(Ax^\gamma+By^\delta\right)\right]d\xi d\eta \, .$$

Now, by virtue of the familiar result (1.23) and subsequently to the condition (1.7), the right hand side of (2.1) is readily verified.

The importance of the result (2.1) lies in the fact that many more interesting double integrals can be evaluated easily by choosing $f(z)$ in convenient form as shown below.

Proof of (2.2): In order to prove the result (2.2), we first set

$$f(z) = z^{\sigma_3}\overline{H}^{m,0}_{p,q}\left[Wz^{\sigma_4}\left|\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,n},\left(a_j,\alpha_j\right)_{n+1,p} \\ \left(b_j,\beta_j\right)_{1,m},\left(b_j,\beta_j;B_j\right)_{m+1,q}\end{matrix}\right.\right]$$

In the equation (2.1) and evaluate the resulting integral as follows:

First express $\overline{H}_1$ in the double contour integral form (1.7), interchange the order integration which is permissible due to absolute convergence of the integrals thus involved in the process, and evaluate the z -integral with the help of (1.24) after using the property (1.26). On interpreting the result thus obtained by virtue of (1.7) we arrive at the right hand side of (2.2).

Proof of (2.3): In order to prove the result (2.3), we first set

$$f(z) = z^{\sigma_3}\psi_{\mu n}\left[c^{\sigma_4/2}, 2W\left(c/z\right)^{-\sigma_4/2}\right]$$

In the result (2.1) and evaluate the resulting integral as follows:

First express the spheroidal function in the expanded form (1.21), change the order of integration and summation which is justified due to the uniform convergence of the series representing spheroidal functions. By expressing the Bessel function thus involved in the form of $\overline{H}$ -function using the result (1.25), we interpret the $\overline{H}_1$ -function in the contour integral form (1.7). Again, change the order of integration by virtue of De La Valle Poussin's well known theorem ([2],p.504) due to absolute convergence of the integrals thus involved. Then, evaluating the inner integral by virtue of result (1.24) and inverting the double contour integrals by definition (1.7), we get the required result (2.3).

Regarding the convergence of the series on the right hand side of (2.3) it would be worth mentioning that the ratio $\frac{a_{r+2}}{a_r}$ is $-\frac{c^2}{4r^2}$, and the ratio of gammas involving r (even or odd) ([6],p.47(4)), hence the series is uniformly and absolutely convergent by M-test.

Special Cases

Since the $\overline{H}$ -function of two variables and spheroidal functions emerge many higher transcendental functions and polynomials, a large number of new and interesting results follow as special cases but we record here only a few of them, for the lack of space.

(i) In (2.2) if we put $A = B, \gamma = \delta = \sigma_4 = 1, s = k = 0, L_j = S_j = 1$ and replace $\frac{\alpha}{\gamma} + \frac{\beta}{\delta} + \sigma_3, W, u, v, \sigma_1, \sigma_2$ respectively by $\lambda, \alpha, \beta, \delta, h, k$, we arrive at the result obtained by Gupta and Mittal ([10],p.12).

(ii) In addition to (i), if we set $p = 0, m = q = 1, b_1 = 1, \sigma_1 = \sigma_2, \alpha_i = A_i (i = 1, 2, ..., p_1), \beta_j = B_j = 1 (j = 1, 2, ..., q_1)$, we arrive at the result obtained by Pathak ([17],p.12) by virtue of relation:

$$\overline{H}_{0,1}^{1,0}\left[ax\Big|_{(0,1)}^{-}\right] = e^{-ax}$$

(iii) Again, if we take $p = q = p_1 = q_1 = 0, A = B = \gamma = \delta = 1, L_j = 1 (j = 1, ..., n_2), S_j = 1 (j = m_2 + 1, ..., q_2)$ our integral formula (2.2) would correspond to a known result due to Panda ([16],p.312, eq. (3.6)).

(iv) If $c \to 0, z \to \infty$ such that (cz) remains finite, (2.3) reduces to the following new result by virtue of (1.22):

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\overline{H}_1\left[ux^s y^k\left(Ax^\gamma + By^\delta\right)^{\sigma_1}, v\left(Ax^\gamma + By^\delta\right)^{\sigma_2}\right]\left(Ax^\gamma + By^\delta\right)^{\sigma_3 - \frac{\sigma_4}{2}\left(\mu + \frac{1}{2}\right)}$$

$$J_{n+\mu+\frac{1}{2}}\left[2W\left(Ax^\gamma + By^\delta\right)^{\sigma_4/2}\right]dxdy = \frac{A^{-\frac{\alpha}{\gamma}}B^{-\frac{\beta}{\delta}}W^{-2\theta}}{\gamma\delta\sigma_4}\overline{H}_{p_1+2,q_1:p_2+2,q_1+1:p_3,q_3}^{0,1:m_2,n_2+2:m_3,n_3}\left[\begin{matrix}MW^{-2\phi}\\ vW^{-\frac{2\sigma_2}{\sigma_4}}\end{matrix}\right.$$

$$\left.\begin{matrix} (a_{p_1},\alpha_{p_1};A_{p_1}),\left(1\pm\frac{1}{2}[1+r+2\mu-2\theta];\phi,\frac{\sigma_2}{\sigma_4}\right),(c_j,\gamma_j;K_j)_{1,n_2},\left(1-\frac{\alpha}{\gamma},\frac{s}{\gamma};1\right),\left(1-\frac{\beta}{\delta},\frac{k}{\delta};1\right),(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_{q_1},\beta_{q_1};B_{q_1}),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{\alpha}{\gamma}-\frac{\beta}{\delta},\frac{s}{\gamma}+\frac{k}{\delta};1\right),(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right] \quad (3.1)$$

Valid under the conditions as given for (2.3).

(v) In (3.1) , if we set $\mu=-\frac{1}{2}, A, B\gamma, \delta, \sigma_2, \sigma_4, A_j(j=1,\ldots,p_1), B_j(j=1,\ldots,q_1), L_j(j=1,\ldots,n_2),$ $S_j(j=m_2+1,\ldots,q_2), F_1, m_3, q_3$ each equal to unity; s,k,v,p_3,n_3,f_1 each equal to zero; and replace $\frac{\alpha}{\gamma}+\frac{\beta}{\delta}+\sigma_3$ by η , we get the result obtained by Gupta and Jain ([8], p.605).

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