ISSN2277-3061

Numerical Solution of System of Two Nonlinear Volterra Integral Equations

Dalal Adnan Maturi[1]
[1]Departement of Mathematics, Faculty of Science, King Abdulaziz University, P.O.Box 42664,Jeddah 21551, Saudi Arabia
Email:maturi_dalal@yahoo.com

ABSTRACT

In this paper, using the implicit trapezoidal rule in conjunction with Newton's method to solve nonlinear system.We have used a Maple 17 program to solve the System of two nonlinear Volterra integral equations. Finally, several illustrative examples are presented to show the effectiveness and accuracy of this method.

Keywords

System of two nonlinear Volterra integral equations; Implicit trapezoidal rule; Newton's method.

Academic Discipline And Sub-Disciplines

Mathematics, Integral Equation and Elasticity.

TYPE (METHOD/APPROACH)

Research Paper.

INTRODUCTION

In this paper, we consider the Volterra integral equation of the second kind

$$x(t) = f(t) + \int_0^t k\big(t, s, x(s)\big)\, ds, \quad (1)$$

Where x, f and k are vector-valued functions with m components. If f and k are continuous and $k(t, s, x(s))$ satisfies a Lipcshitz condition with respect to x, then a unique solution $x(t)$ of (1) exists[1,4,7].

Volterra integral equations have been found to be effective to describe some application such as population dynamics, renewal equations, nuclear reactor dynamics, viscoelasticity, study of epidemics, super fluidity, damped vibrations, heat conduction and diffusion [1,7].

In this paper, we present the computation of numerical solution of system of two nonlinear Volterra integral equation of the second kind.

PRELIMINARIES

In this section, we recall the main theorems [7].

Theorem 1. Consider the equation

$$x(t) = f(t) + \int_0^t p(t, s) k(t, s) x(s)\, ds \quad (2)$$

Where

1) $f(t)$ is continuous in $0 \le t \le T$.
2) $k(t, s)$ is a continuous function in $0 \le s \le t \le T$,
3) for each continuous function h and all $0 \le \tau_1 \le \tau_2 \le t$ the integrals

$$\int_{\tau_1}^{\tau_2} p(t, s) k(t, s) h(s)\, ds$$

$$\int_0^t p(t, s) k(t, s) h(s)\, ds$$

are continuous functions of t ,
4) $p(t, s)$ is absolutely integrable with respect to s for all $0 \le t \le T$,
5) there exist points $0 = T_0 < T_1 < T_2 < \cdots < T_N = T$ such that with $t \ge T_1$

$$k \int_{T_i}^{\min(t, T_{i+1})} |p(t, s)|\, ds \le \alpha < 1,$$

Were $k = \max_{0 \le s \le t \le T} |k(t, s)|$,

6) for every $t \ge 0$ such that with $t \ge T_1$

$$\lim_{\delta \to 0^+} \int_t^{t+\delta} |p(t + \delta, s)|\, ds = 0.$$

Then (2) has a unique continuous solution in $0 \le t \le T$.

Theorem 2. Consider the equation

$$x(t) = f(t) + \int_0^t p(t, s) k(t, s, x(s))\, ds \quad (3)$$

Where

1) $f(t)$ is continuous in $0 \le t \le T$.
2) $k(t, s, u)$ is a continuous function in $0 \le s \le t \le T$, $-\infty < u < \infty$,
3) the Lipschitz condition $|k(t, s, y) - k(t, s, z)| \le L|y - z|$ is satisfies for $0 \le s \le t \le T$ and all y and z,
4) $p(t, s)$ satisfies conditions (3)-(4) of Theroem 1 with k replaced by L and $k(t, s, h(s))$ instead of $k(t, s)h(s)$.

Then (3) has a unique continuous solution in $0 \le t \le T$.

ISSN2277-3061

THE MATHEMATICSOF THE VOLTERRA PROCEDURE

In this section, we use the technique of the Volterraeqution [2,7] to find an approximates the solution $x(t)$ of (1) at the equally spaced points $t_n = t_0 + nh$ for $n = 1, \cdots, N$ where $t_0 = 0$ and $N$is the total number of steps of size $h. X_n$ denotes the approximation of $x(t)$ at $t = t_n$.

Setting $t = t_n$ in (1), we have

$$x(t_n) = f(t_n) + \int_0^{t_n} k(t_n, t, x(t))\, dt \tag{4}$$

By the composite trapezoidal rule an approximation of the integral in (4) is

$$\frac{h}{2}\left[k\big(t_n, t_0, x(t_0)\big) + 2\sum_{j=1}^{n-1} k(t_n, t_j, x(t_j)) + k(t_n, t_n, x(t_n))\right] \tag{5}$$

Replacing $x(t_n)$ in (4) and (5) by X_n, we obtain the implicit trapezoidal rule

$$X_n = f(t_n) + h\left[\frac{1}{2}k(t_n, t_0, X_0) + \sum_{j=1}^{n-1} k(t_n, t_j, X_j) + \frac{1}{2}k(t_n, t_n, X_n)\right] \tag{6}$$

Where $X_0 = f(0)$since $x(0) = f(0)$.
Defining σ_n by

$$\sigma_n = f(t_n) + h\left[\frac{1}{2}k(t_n, t_0, X_0) + \sum_{j=1}^{n-1} k(t_n, t_j, X_j)\right] \tag{7}$$

We can rewrite (6) as

$$X_n - \frac{1}{2}hk(t_n, t_n, X_n) - \sigma_n = 0, \tag{8}$$

Where 0denotes the zero vector. From (8), we see that X_n is the solution of the vector equation

$$\phi(u) = 0, \tag{9}$$

Where ϕ is the vector-valued function

$$\phi(u) = u - \frac{1}{2}hk(t_n, t_n, u) - \sigma_n \tag{10}$$

We will obtain an approximation to the solution $X_n$of (9) by way of the matrix-valued function Gdefined in (11). If $A(u)$ is an m by m matrix-valued function that is invertible in a neighborhood of X_n, then X_n is a fixed point of

$$G(u) = u - A(u)\phi(u). \tag{11}$$

Assuming the components of $G(u)$ have continuous first and second order partial derivatives and that the first order partial derivatives and that the first order partial derivatives at
$X_n$are equal to zero, it can be shown that if $A(u)$ is set equal to the Jacobian matrix of the function ϕ, the iterates $X_n^{(p)}$ defined by (13) below will usually converge quadratically to X_n provided the starting value is sufficiently close to X_n. The Jacobian matrix of ϕ is the m by m matrix $J(u)$ with the element

$$J(u)_{ij} = \frac{\partial}{\partial u_j}\phi_i(u) = \delta_{ij} - \frac{1}{2}h\frac{\partial}{\partial u_j}k_i(t_n, t_n, u) \tag{12}$$

In row i and column, where δ_{ij} is the Kronecker delta. Details of the statements made here follow from the discussion of Newton's method for nonlinear systems in [2]. Linz gives a brief outline of the trapezoidal rule and Newton's method for Volterra integral systems of the second kind in Section of [7].

We obtain $X_n$from $X_{n-1}$by setting $X_n^{(0)} = X_{n-1}$and then generating the iterates $X_n^{(p)}$ from

$$X_n^{(p)} = G\left(X_n^{(p-1)}\right) = X_n^{(p-1)} - J^{-1}\left(X_n^{(p-1)}\right)\phi\left(X_n^{(p-1)}\right) \tag{13}$$

For $p = 1,2,3,\cdots$. (This is Newton's method for nonlinear systems.) Let y denote the solution of the matrix equation

$$J\left(X_n^{(p-1)}\right)y = \phi\left(X_n^{(p-1)}\right). \tag{14}$$

Then the iteration formula (13) becomes

$$X_n^{(p)} = X_n^{(p-1)} - y. \tag{15}$$

We compute the solution $y = J^{-1}\left(X_n^{(p-1)}\right)\phi\left(X_n^{(p-1)}\right)$ using the command Linear Solve. The iterates $X_n^{(p)}$ are computed until the infinity norm of the vector y is less than a prescribed tolerance Tol. Then X_n is assigned the value of the last iterate [2,7].

NUMERICAL EXAMPLES

In this section, we solve some examples, and we can compare the numerical results with the exact solution.

Example1. Consider the system of Volterra integral equations

$$\mathrm{X}_1(t) = 1 - \frac{t^2}{2} + \int_0^t (\mathrm{X}_1(s) + se^s\mathrm{X}_2(s))ds$$

$$\mathrm{X}_2(t) = 1 + \frac{t^2}{2} + \int_0^t (-se^{-s}\,\mathrm{X}_1(s) - \mathrm{X}_2(s))ds$$

With the exact solution $\mathrm{X}_1(t) = e^t$ and $\mathrm{X}_2(t) = e^{-t}$.

Table.1 Numerical results and exact solution of systems of Two Nonlinear Volterra integral equations for example 1.

t	$X_1(t)$	$X_2(t)$	$Exact1 = e^t$	$Exact2 = e^{-t}$
0.0	1.00000	1.00000	1.00000	1.00000
0.1	1.10526	0.90476	1.10517	0.90484
0.2	1.22160	0.81859	1.22140	0.81873
0.3	1.35019	0.74063	1.34986	0.74082
0.4	1.49230	0.67008	1.49182	0.67032
0.5	1.64937	0.60625	1.64872	0.60653
0.6	1.82295	0.54849	1.82212	0.54881
0.7	2.01480	0.49622	2.01375	0.49659
0.8	2.22682	0.44892	2.22554	0.44933
0.9	2.46113	0.40612	2.45960	0.40657
1.0	2.72007	0.36738	2.71828	0.36738
1.1	3.00623	0.33233	3.00417	0.33287
1.2	3.32245	0.30060	3.32012	0.30119
1.3	3.67190	0.27189	3.66930	0.27253
1.4	4.05804	0.24590	4.05520	0.24660
1.5	4.48474	0.22239	4.48169	0.22313

ISSN2277-3061

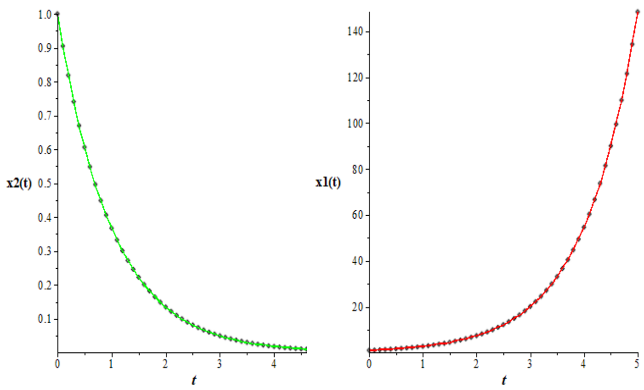

Fig. 1 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 1..

Example2. Consider the system of Volterra integral equations

$$X_1(t) = 1 + t^2 - \frac{t^3}{3} - \frac{t^4}{3} + \int_0^t ((t-s)^3 X_1(s) + (t-s)^2 X_2(s))ds$$

$$X_2(t) = 1 - t - t^3 - \frac{t^4}{4} - \frac{t^5}{4} - \frac{t^7}{420} + \int_0^t ((t-s)^4 X_1(s) + (t-s)^3 X_2(s))ds$$

With the exact solution $X_1(t) = 1 + t^2$ and $X_2(t) = 1 + t - t^3$.

Table.2 Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 2.

t	$X_1(t)$	$X_2(t)$	$Exact1 = 1 + t^2$	$Exact2 = 1 + t - t^3$
0.0	1.00000	1.00000	1.00000	1.00000
0.1	1.01018	0.89903	1.01000	1.09900
0.2	1.04020	0.79210	1.04000	1.19200
0.3	1.08945	0.67309	1.09000	1.27300
0.4	1.15693	0.53564	1.16000	1.33600
0.5	1.24125	0.37302	1.25000	1.37500
0.6	1.34060	0.17802	1.36000	1.38400
0.7	1.45276	-0.05716	1.49000	1.35700
0.8	1.57509	-0.34107	1.64000	1.28800
0.9	1.70450	-0.68310	1.81000	1.17100
1.0	1.83743	-1.09365	2.00000	1.00000
1.1	1.96978	-1.58430	2.21000	0.76900
1.2	2.09686	-2.16793	2.44000	0.47200
1.3	2.21327	-2.85900	2.69000	0.10300
1.4	2.31281	-3.67372	2.96000	-0.34400
1.5	2.38830	-4.63041	3.25000	-0.87500

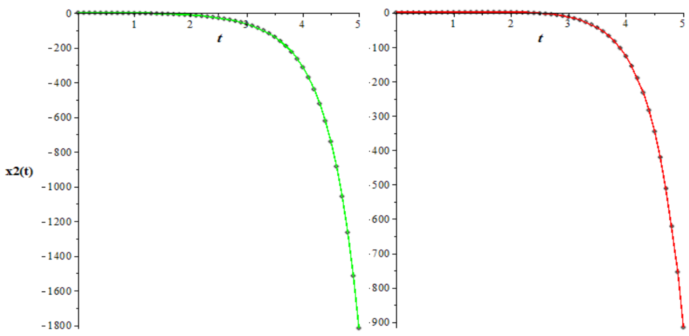

Fig. 2 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 2.

Example3. Consider the system of Volterra integral equations

$$X_1(t) = \cos t - t\sin t + \int_0^t (\sin(t-s)\,X_1(s) + \cos(t-s)\,X_2(s))ds$$

$$X_2(t) = \sin t - t\sin t + \int_0^t (\cos(t-s)X_1(s) - \sin(t-s)\,X_2(s))ds$$

With the exact solution $X_1(t) = \cos t$ and $X_2(t) = \sin t$.

Table.3 Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 3.

t	$X_1(t)$	$X_2(t)$	$Exact1 = \cos t$	$Exact2 = \sin t$
0.0	1.00000	0.00000	1.00000	0.00000
0.1	0.99949	0.18957	0.99500	0.09983
0.2	0.99687	0.35534	0.98007	0.19867
0.3	0.98990	0.49396	0.95534	0.29552
0.4	0.97620	0.60260	0.92106	0.38942
0.5	0.95336	0.67899	0.87758	0.47943
0.6	0.91893	0.72143	0.82534	0.56464
0.7	0.87051	0.72884	0.76484	0.64422
0.8	0.80577	0.70076	0.69671	0.71736
0.9	0.72254	0.63737	0.62161	0.78333
1.0	0.61883	0.53949	0.54030	0.84147
1.1	0.49289	0.40857	0.45360	0.89121
1.2	0.34324	0.24665	0.36236	0.93204
1.3	0.16875	0.05636	0.26750	0.96356
1.4	-0.03138	-0.15913	0.16997	0.98545
1.5	-0.25752	-0.39615	0.07074	0.99749

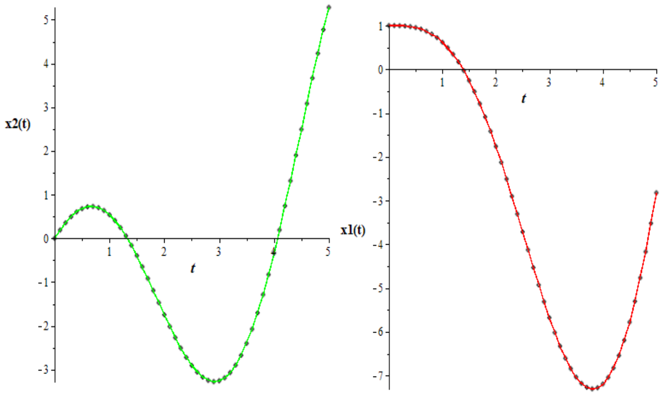

Fig. 3 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 3.

Example4. Consider the system of Volterra integral equations

$$X_1(t) = e^t - 2t + \int_0^t (e^{-s} X_1(s) + e^s X_2(s))ds$$

$$X_2(t) = e^{-t} + \sinh(2t) + \int_0^t (e^s X_1(s) + e^{-s} X_2(s))ds$$

With the exact solution $X_1(t) = e^t$and $X_2(t) = e^{-t}$.

Table.4 Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 4.

t	$X_1(t)$	$X_2(t)$	$Exact1 = e^t$	$Exact2 = e^{-t}$
0.00	1.00000	1.00000	1.00000	1.00000
0.01	1.01025	1.03025	1.01005	0.99005
0.02	1.02103	1.06102	1.02020	0.98020
0.03	1.03233	1.09232	1.03045	0.97045
0.04	1.04419	1.12417	1.04081	0.96079
0.05	1.05662	1.15658	1.05127	0.95123
0.06	1.06965	1.18957	1.06184	0.94176
0.07	1.08328	1.22316	1.07251	0.93239
0.08	1.09754	1.25737	1.08329	0.92312
0.09	1.11246	1.29221	1.09417	0.91393
0.10	1.12805	1.32771	1.10517	0.90484
0.11	1.14434	1.36389	1.11628	0.89583
0.12	1.16134	1.40076	1.12750	0.88692
0.13	1.17909	1.43835	1.13883	0.87810
0.14	1.19760	1.47669	1.15027	0.86936
0.15	1.21690	1.51578	1.16183	0.86071

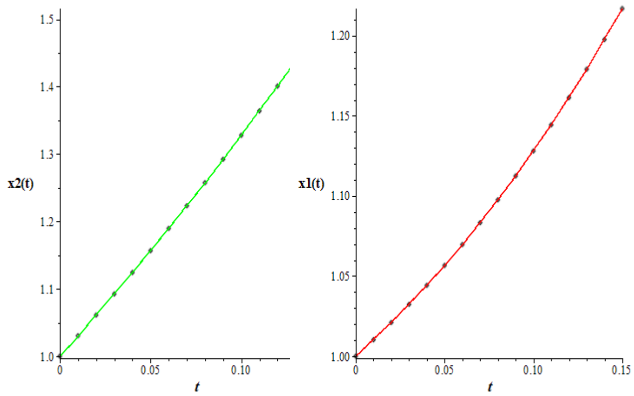

Fig. 4 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 4.

Example5. Consider the system sofVolterra integral equations

$$X_1(t) = \sec^2 t - 2\tan t + t + \int_0^t (X_1(s)+X_2(s))ds$$

$$X_2(t) = \tan^2 t - t + \int_0^t (X_1(s)-X_2(s))ds$$

With the exact solution $X_1(t) = \sec^2 t$and $X_2(t) = \tan^2 t$.

Table.5 Numerical results and exact solution of systems of two Nonlinear Volterra integral equations for example 5.

t	$X_1(t)$	$X_2(t)$	$Exact1 = \sec^2 t$	$Exact2 = \tan^2 t$
0.00	1.00000	0.00000	1.00000	0.00000
0.01	1.00010	0.00010	1.00010	0.00010
0.02	1.00040	0.00040	1.00040	0.00040
0.03	1.00090	0.00090	1.00090	0.00090
0.04	1.00160	0.00160	1.00160	0.00160
0.05	1.00251	0.00250	1.00250	0.00250
0.06	1.00361	0.00361	1.00361	0.00361
0.07	1.00492	0.00492	1.00492	0.00492
0.08	1.00643	0.00643	1.00643	0.00643
0.09	1.00815	0.00814	1.00814	0.00814
0.10	1.01007	0.01007	1.01007	0.01007
0.11	1.01220	0.01220	1.01220	0.01220
0.12	1.01454	0.01454	1.01454	0.01454
0.13	1.01710	0.01709	1.01709	0.01709
0.14	1.01986	0.01986	1.01986	0.01986
0.15	1.02285	0.02284	1.02284	0.02284

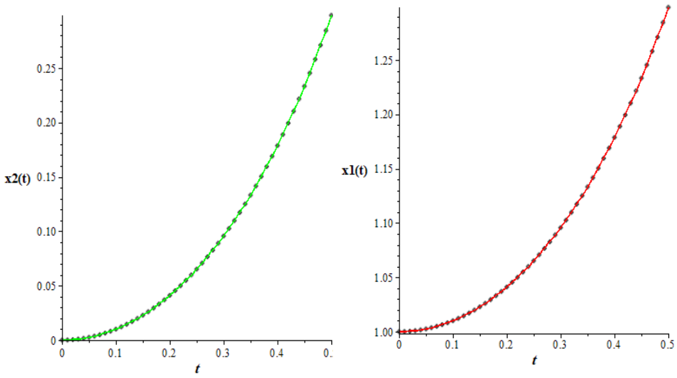

Fig. 5 The exact and approximate solutions result of systems of two Nonlinear Volterra integral equations for example 5.

Conclusion

In this paper, we compute the numerical solution of some examples and compare it with their exact solution. The computed values and graphics, illustrated by the results, agree well with the exact solution.

REFERENCES

[1] A.M.Wazwaz,, Linear and Nonlinear Integral Equation: Methods and Applications, Springer, 2011.

[2] F. Mirzaee, Numerical Computional Solution of the Linear Volterra Integral Equations System Via Rationalized Haar Functions, Journal of King Saud University- Science, 22(4)(2010),265-268.

[3] J. Biazar and M. Eslami, Modified HPM for Solving Systems of Volterra Integral Equation of the Second Kind, Journal of King Saud University-Science, 23(1)(2011),35-39.

[4] L.C. Becker and M. Wheeler, Numerical and Graphical Solutions of Volterra Equations of the Second Kind,Maple Application Center, 2005.

[5] M.I. Berenguer, D. Gamez, A. I. Garralda-Guillem,M. Ruiz Galan, and M.C. Serrano Perez, Biorthogonal Systems for Solving Volterra Integral equation Systems of the Second Kind, Journal of Computional and Applied Mathematics,235(7)(2011),1875-1883.

[6] P. Linz, Analytical and Numerical Methods for Volterra Equations, Studies in Applied Mathematics 7, SIAM, Philadelphia, 1985.

[7] R.L.Burden and J. Douglas Faires, Numerical Analysis, 8th ed., Thomson Brooks/Cole, Belmont, CA, 2005.

[8] T. A. Burton, Volterra Integral and Differential Equations, 2nd ed., Mathematics in Science & Engineering, 202, Elsevier, 2005.

[9] V. Balakumar, K. Murugesan, Numerical Solution of Systems of Linear Volterra Integral Equations Using Block-Pulse Functions, Malaya Journal of Matematik, S(1)(2013)77-84.