ISSN 2277-3061

Three dimensional surfaces foliated by the equiform motion of pseudohyperbolic surfaces in E^7

E. M. Solouma [1]
Department of Mathematics and Statistics, College of Science,
Al Imam Muhammad Ibn Saud Islamic University, Kingdom of Saudi Arabia
E-mail: emadms74@yahoo.com.
Department of Mathematics, College of Science, Beni-Suef University, Beni-Suef, Egypt

Abstract

In this paper we study three dimensional surfaces in E^7 generated by equiform motions of a pseudohyperbolic surface. The properties of these surfaces up to the first order are investigated. We prove that three dimensional surfaces in E^7 in general, is contained in a canal hypersurface, which is gained as envelope of a one-parametric set of 6-dimensional pseudohyperbolic. Finally we give an example.

Mathematical Subject Classification (2010): 53A05, 53A17, 53B30.

Keywords: Equiform motion; hypersurfaces; Lorentz metrics.

Council for Innovative Research
Peer Review Research Publishing System
Journal: INTERNATIONAL JOURNAL OF COMPUTERS & TECHNOLOGY
Vol 12, No. 9
editor@cirworld.com
www.cirworld.com, www.ijctonline.com

1. Introduction

An equiform transformation in the n-dimensional Euclidean space $\mathbb{R}^n$ is an affine transformation whose linear part is composed from an orthogonal transformation and a homothetical transformation. Such an equiform transformation maps points $\mathbf{x} \in \mathbb{R}^n$ according to

$$\mathbf{x} \mapsto sA\mathbf{x} + \mathbf{d}, \quad .5cmA \in SO(n), s \in \mathbb{R}^+, \mathbf{d} \in \mathbb{R}^n. \tag{1}$$

The number s is called the scaling factor. An equiform motion is defined if the parameters of (1), including s, are given as functions of a time parameter t. Then a smooth one-parameter equiform motion moves a point $\mathbf{x}$ via $\mathbf{x}(t) = s(t)A(t)\mathbf{x}(t) + \mathbf{d}(t)$. The kinematic corresponding to this transformation group is called equiform kinematic. See [2]. Recently, the equiform kinematic geometry has been used in computer vision and reverse engineering of geometric models such as the problem of reconstruction of a computer model from an existing object which is known (a large number of) data points on the surface of the technical object [9, 11]. In [8], they studied two-parameter spatial motions $M_2(\lambda, \mu)$ in three dimensional Euclidean space from a differential geometric point of view, which (up to the second order) instantaneously move on locally one-dimensional point paths. In [1, 12], they studied some first order properties of cyclic surfaces generated by the equiform motions in five dimensional Euclidian space and semi-Euclidean space.

In Minkowski (semi-Euclidean) space E^3 with scalar product $<x, y> = -x_1y_1 + x_2y_2 + x_3y_3$ the pseudosphere or Lorentz sphere and the pseudohyperbolic surface play the same role as sphere in Euclidean space. Lorentz sphere of radius $r > 0$ in E^3 is the quadric

$$S^2(r) = \{p \in E^3 :< p, p >= r^2\}.$$

This surface is timelike and is the hyperboloid of one sheet $-x_1^2 + x_2^2 + x_3^2 = r^2$ which is obtained by rotating the hyperbola $-x_1^2 + x_3^2 = r^2$ in the plane $x_2 = 0$ with respect to the x_1-axis. The pseudohyperbolic surface is the quadratic

$$H_0^2(r) = \{p \in E^3 :< p, p >= -r^2\}.$$

This surface is spacelike and is the hyperboloid of two sheet $-x_1^2 + x_2^2 + x_3^2 = -r^2$ which is obtained by rotating the hyperbola $x_1^2 - x_3^2 = r^2$ in the plane $x_2 = 0$ with respect to the x_1-axis [10].

In this paper we consider the equiform motions of a pseudohyperbolic surface $k_\circ$ in E^n. The point paths of the pseudohyperbolic surface, generate three-dimensional surface, contains the positions of the starting pseudohyperbolic surface $k_\circ$. The first order properties of these surfaces for the points of these pseudohyperbolic surfaces have been studied for arbitrary dimensions $n \geq 3$. We restrict our considerations to dimension $n = 7$ because, at any moment the infinitesimal transformations of the motion maps the points of the pseudohyperbolic surface $k_\circ$ to the velocity vectors, whose end points will form an affine image of $k_\circ$ (in general a pseudohyperbolic surface $k_\circ$). Both these surfaces are space and therefore span a subspace W of E^n with $n \leq 7$. Moreover, we show that any three-dimensional surfaces in E^7 is in general contained in a canal hypersurface, which is gained as envelope of a one-parametric set of 6-dimensional pseudosphere.

2. Local study in canonical frames

Consider a unit pseudohyperbolic surface $k_\circ$ in the space $\pi_\circ = [x_1x_2x_3]$ centered at the origin represented by

$$x(\theta,\phi) = (\cosh\theta, \sinh\theta\sin\phi, \sinh\theta\cos\phi, 0,0,0,0)^T, \theta \in \mathbb{R} \, and \, \phi \in [0,2\pi],$$

the general representation of the motion of three-dimensional surface in E^7 foliated by two-dimensional pseudohyperbolic surface is given by

$$X(t,\theta,\phi)=s(t)A(t)x(\theta,\phi)+d(t), t\in \mathbb{R} \quad (2)$$

where $d(t)=(b_1(t),b_2(t),b_3(t),b_4(t),b_5(t),b_6(t),b_7(t))^T$ describes the position of the origin of Σ° at the time t, $A(t)=(a_{ij}(t))$, $1\le i,j\le 7$ is a semi orthogonal matrix and $s(t)$ provides the scaling factor of the moving system. Moreover we assume that all involved functions are of class C^1. Using Taylor's expansion, up to the first order then the representation of the motion is given by

$$X(t,\theta,\phi)=\{s(0)A(0)+[\dot{s}(0)A(0)+s(0)\dot{A}(0)]t\}x(\theta,\phi)+d(0)+t\dot{d}(0),$$

where $(\cdot)$ denotes differentiation with respect to time ($t=0$). As an equiform motion has an invariant point, we can assume without loss of generality that the moving frame E^7 and fixed frame Σ coinciding at the zero position $(t=0)$, then we have

$$A(0)=I,\quad s(0)=1 \quad and \quad d(0)=0,$$

thus

$$X(t,\theta,\phi)=[I+(s'I+\Omega)t]x(\theta,\phi)+td',$$

where $\Omega=\dot{A}(0)=(\omega_k), k=1,2,3,...,21$ is a semi skew symmetric matrix. In this paper all values of s, b_i and their derivatives are computed at $t=0$ and for simplicity, we write s' and b_i' instead of $\dot{s}(0)$ and $\dot{b}_i(0)$ respectively. In these frames, the representation of $\mathbf{X}(t,\theta,\phi)$ is given by

$$\begin{pmatrix} X_1\\ X_2\\ X_3\\ X_4\\ X_5\\ X_6\\ X_6 \end{pmatrix}=\begin{pmatrix} 1+s't & t\omega_1 & t\omega_2 & t\omega_3 & t\omega_4 & t\omega_5 & t\omega_6\\ t\omega_1 & 1+s't & t\omega_7 & t\omega_8 & t\omega_9 & t\omega_{10} & t\omega_{11}\\ t\omega_2 & -t\omega_7 & 1+s't & t\omega_{12} & t\omega_{13} & t\omega_{14} & t\omega_{15}\\ t\omega_3 & -t\omega_8 & -t\omega_{12} & 1+s't & t\omega_{16} & t\omega_{17} & t\omega_{18}\\ t\omega_4 & -t\omega_9 & -t\omega_{13} & -t\omega_{16} & 1+s't & t\omega_{19} & t\omega_{20}\\ t\omega_5 & -t\omega_{10} & -t\omega_{14} & -t\omega_{17} & -t\omega_{19} & 1+s't & t\omega_{21}\\ t\omega_6 & -t\omega_{11} & -t\omega_{15} & -t\omega_{18} & -t\omega_{20} & -t\omega_{21} & 1+s't \end{pmatrix}\begin{pmatrix} \cosh\phi\\ \sinh\theta\sin\phi\\ \sinh\theta\cos\phi\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}+t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5'\\ b_6'\\ b_7' \end{pmatrix},$$

or in the equivalent form

$$\begin{pmatrix} X_1\\ X_2\\ X_3\\ X_4\\ X_5\\ X_6\\ X_7 \end{pmatrix}=t\begin{pmatrix} b_1'\\ b_2'\\ b_3'\\ b_4'\\ b_5'\\ b_6'\\ b_7' \end{pmatrix}+\begin{pmatrix} 1+s't\\ t\omega_1\\ t\omega_2\\ t\omega_3\\ t\omega_4\\ t\omega_5\\ t\omega_6 \end{pmatrix}\cosh\theta+\begin{pmatrix} t\omega_1\\ 1+s't\\ -t\omega_7\\ -t\omega_8\\ -t\omega_9\\ -t\omega_{10}\\ -t\omega_{11} \end{pmatrix}\sinh\theta\sin\phi+\begin{pmatrix} t\omega_2\\ t\omega_7\\ 1+s't\\ -t\omega_{12}\\ -t\omega_{13}\\ -t\omega_{14}\\ -t\omega_{15} \end{pmatrix}\sinh\theta\cos\phi \quad (3)$$

$$=t\vec{b}+\vec{a}_\circ\cosh\theta+\vec{a}_1\sinh\theta\sin\phi+\vec{a}_2\sinh\theta\cos\phi.$$

For any fixed t in the above expression (3), we generally gain an elliptical hyperboloid for $\theta \in \mathbb{R}$ and $\phi \in [0, 2\pi]$ centered at the point $t(b'_1, b'_2, b'_3, b'_4, b'_5, b'_6, b'_7)$. The latter elliptical hyperboloid turns to a two-dimensional pseudohyperbolic surface if $\vec{a}_\circ$, $\vec{a}_1$ and $\vec{a}_2$ form an orthogonal basis. This gives the conditions

$$\begin{aligned}\omega_2\omega_7 + \omega_3\omega_8 + \omega_4\omega_9 + \omega_5\omega_{10} + \omega_6\omega_{11} &= -\omega_1\omega_7 + \omega_3\omega_{12} + \omega_4\omega_{13} + \omega_5\omega_{14} + \omega_6\omega_{15} \\ &= -\omega_1\omega_2 + \omega_8\omega_{12} + \omega_9\omega_{13} + \omega_{10}\omega_{14} + \omega_{11}\omega_{15} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2 &= \omega_1^2 - \omega_7^2 - \omega_8^2 - \omega_9^2 - \omega_{10}^2 - \omega_{11}^2 \\ &= \omega_2^2 - \omega_7^2 - \omega_{12}^2 - \omega_{13}^2 - \omega_{14}^2 - \omega_{15}^2 \\ &= a,\end{aligned}$$

where $a \in \mathbb{R}^+$. Thus we get the following equation of the pseudohyperbolic space

$$\sum_{i=1}^{7} \varepsilon_i (x_i - tb'_i)^2 = at^2 - (1 + s't)^2,$$

where $\varepsilon_1 = -1, \varepsilon_j = 1, j = 2,3,4,5,6,7$. The orthogonal projection of these elliptical hyperboloid ($t = const$. in (3)) on the space of the starting pseudohyperbolic surface $\pi_\circ = [x_1 x_2 x_3]$, is

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = t\begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} + \begin{pmatrix} 1+s't \\ t\omega_1 \\ t\omega_2 \end{pmatrix}\cosh\theta + \begin{pmatrix} t\omega_1 \\ 1+s't \\ -t\omega_7 \end{pmatrix}\sinh\theta\sin\phi + \begin{pmatrix} t\omega_2 \\ t\omega_7 \\ 1+s't \end{pmatrix}\sinh\theta\cos\phi. \tag{4}$$

This equation generalizes in five dimension that happens for $\phi = 0$. Namely, if $\phi = 0$ the orthogonal projection of the elliptical hyperboloid in equation (4) on the space $[x_1 x_3]$ is

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} = t\begin{pmatrix} b'_1 \\ b'_3 \end{pmatrix} + \begin{pmatrix} 1+s't \\ t\omega_2 \end{pmatrix}\cosh\theta + \begin{pmatrix} t\omega_2 \\ 1+s't \end{pmatrix}\sinh\theta.$$

This gives Lorentzian circles centered at (tb'_1, tb'_3) and radii by $\sqrt{|t^2\omega_2^2 - (1+s't)^2|}$.

Corollary 2.1

The projection of the ruled surface of tangent to $k_\circ$ into the original space will give a three-dimensional surface in $\mathbb{E}^3$, which is foliated by elliptical hyperboloids. Now from (4) we have

$$X(t,\theta,\phi) = \begin{pmatrix} 1+s't & t\omega_1 & t\omega_2 \\ t\omega_1 & 1+s't & t\omega_7 \\ t\omega_2 & -t\omega_7 & 1+s't \end{pmatrix}\begin{pmatrix} \cosh\theta \\ \sinh\theta\sin\phi \\ \sinh\theta\cos\phi \end{pmatrix} + t\begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix},$$

and the first partial derivatives are

$$X_t = \begin{pmatrix} b_1' \\ b_2' \\ b_3' \end{pmatrix} + \begin{pmatrix} s' & \omega_1 & \omega_2 \\ \omega_1 & s' & \omega_7 \\ \omega_2 & -\omega_7 & s' \end{pmatrix} \begin{pmatrix} \cosh\theta \\ \sinh\theta\sin\phi \\ \sinh\theta\cos\phi \end{pmatrix},$$

$$X_\theta = (\sinh\theta, \cosh\theta\sin\phi, \cosh\theta\cos\phi)^T,$$

$$X_\phi = (0, \sinh\theta\cos\phi, -\sinh\theta\sin\phi)^T.$$

Then the linearly dependent points

$$\sinh\theta[-s' - b_1'\cosh\theta + b_2'\sinh\theta\sin\phi + b_3'\sinh\theta\cos\phi] = 0,$$

we get

$$\sinh\theta[-s' + \langle d', x(\theta,\phi)\rangle] = 0.$$

The latter equation characterizes the instantaneous curve of contact.

3.Tangent pseudosphere of three-dimensional surface in E^7

In this section we will how that at any instant t there exists a pseudosphere $K(t)$, which is tangent to a given three-dimensional surface (2) in all points of the instantaneous position $k(t)$ of the pseudohyperbolic surface $k_\circ$. Without loss of generality we investigate the situation at the zero position. Any pseudosphere $K_\circ$ which is tangent to the given three-dimensional surface (2) along $k_\circ$ has to contain $k_\circ$, hence the center of $K_\circ$ has coordinates $(0,0,0,m_4,m_5,m_6,m_7)$ with $m_4,m_5,m_6,m_7 \in \mathbb{R}$. On the other hand since $K_\circ$ has to be tangent to all velocity vectors of the motion, the center of $K_\circ$ has to lie in each of the hyperplanes through the points of $k(t)$ orthogonal to these velocity vectors. This gives us the additional condition

$$\begin{aligned} & m_4(b_4' + \omega_3\cosh\theta - \omega_8\sinh\theta\sin\phi - \omega_{12}\sinh\theta\cos\phi) \\ & + m_5(b_5' + \omega_4\cosh\theta - \omega_9\sinh\theta\sin\phi - \omega_{13}\sinh\theta\cos\phi) \\ & + m_6(b_6' + \omega_5\cosh\theta - \omega_{10}\sinh\theta\sin\phi - \omega_{14}\sinh\theta\cos\phi) \\ & + m_7(b_7' + \omega_6\cosh\theta - \omega_{11}\sinh\theta\sin\phi - \omega_{15}\sinh\theta\cos\phi) \\ & = -s' - b_1'\cosh\theta + b_2'\sinh\theta\sin\phi + b_3'\sinh\theta\cos\phi. \end{aligned} \tag{5}$$

By comparing the coefficients of $\{1, \cosh\theta, \sinh\theta\sin\phi, \sinh\theta\cos\phi\}$ in (5), we have the system of linear equations

$$BM = H, \tag{6}$$

where

$$B = \begin{pmatrix} b_4' & b_5' & b_6' & b_7' \\ \omega_3 & \omega_4 & \omega_5 & \omega_6 \\ \omega_8 & \omega_9 & \omega_{10} & \omega_{11} \\ \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} \end{pmatrix}, \quad M = \begin{pmatrix} m_4 \\ m_5 \\ m_6 \\ m_7 \end{pmatrix} \text{and} \quad H = \begin{pmatrix} -s' \\ -b_1' \\ -b_2' \\ -b_3' \end{pmatrix}.$$

If B is a regular matrix, we get

$$M = B^{-1}H. \tag{7}$$

Therefore, we have the following theorem:

Theorem 3.1

Definition 3.1 *Canal hypersurfaces in* $\mathbb{E}^n$ *are envelope hypersurfaces of one-parametric sets of pseudospheres.*

Therefore, we have the following theorem

Theorem 3.2

3.1 The singular cases

If the system of equations (6) is singular, we have many cases:

Case 1. $rank(B) = rank(B \setminus H) = 3$. In this case, we have a one-parametric set of pseudospheres whose centers fulfil a straight line in the $x_4x_5x_6x_7 - space$

$$M = (0,0,0,m_4, x_5(m_4), x_6(m_4), x_7(m_4)),$$

where

$$x_5(m_4) = \frac{1}{\Delta^*}[(\omega_5\omega_{11} - \omega_6\omega_{10})(s' + b_4'm_4) + (b_6'\omega_{11} - b_7'\omega_{10})(b_1' + \omega_3 m_4)$$
$$+ (b_7'\omega_5 - b_6'\omega_6)(b_2' + \omega_8 m_4)],$$

$$x_6(m_4) = \frac{1}{\Delta^*}[(\omega_6\omega_9 - \omega_4\omega_{11})(s' + b_4'm_4) + (b_7'\omega_9 - b_5'\omega_{11})(b_1' + \omega_3 m_4)$$
$$+ (b_5'\omega_6 - b_7'\omega_4)(b_2' + \omega_8 m_4)],$$

$$x_7(m_4) = \frac{1}{\Delta^*}[(\omega_4\omega_{10} - \omega_5\omega_9)(s' + b_4'm_4) + (b_5'\omega_{10} - b_6'\omega_9)(b_1' + \omega_3 m_4)$$
$$+ (b_6'\omega_4 - b_5'\omega_5)(b_2' + \omega_8 m_4)],$$

where

$$\Delta^* = b_5'(\omega_5\omega_{11} - \omega_6\omega_9) + b_6'(\omega_6\omega_9 - \omega_4\omega_{11}) + b_7'(\omega_4\omega_{10} - \omega_5\omega_9),$$

with arbitrary $m_4 \in \mathbb{R}$. Thus, we get a straight line of possible centers.

Case 2. $rank(B) = rank(B \setminus H) = 2$. In this case, we have a two-parametric set of pseudospheres whose centers fulfil a surface in $x_4x_5x_6x_7 - space$

$$M = (0,0,0,m_4, m_5, x_6(m_4,m_5), x_7(m_4,m_5)),$$

where

$$x_6(m_4,m_5) = \frac{m_4(b_6'\omega_3 - b_4'\omega_5) + m_5(b_6'\omega_4 - b_5'\omega_5) + (s'\omega_5 - b_1'b_6')}{b_7'\omega_5 - b_6'\omega_6},$$

$$x_7(m_4,m_5)=\frac{m_4(b_4'\omega_6-b_7'\omega_3)+m_5(b_5'\omega_6-b_7'\omega_4)-(s'\omega_6-b_1'b_7')}{b_7'\omega_5-b_6'\omega_6},$$

with arbitrary $m_4, m_5 \in \mathrm{R}$. Thus, we get a surface of possible centers.

Case 3. $rank(B)=rank(B\setminus H)=1$. In this case, we have a hyperplane of possible centers. **Case 4.** $rank(B)=3\neq rank(B\setminus H)$. In this case we assume

$$\frac{\omega_8}{\omega_{12}}=\frac{\omega_9}{\omega_{13}}=\frac{\omega_{10}}{\omega_{14}}=\frac{\omega_{11}}{\omega_{15}}=\lambda,\quad \frac{b_2'}{b_3'}\neq\lambda.$$

By using the homogenous coordinates

$$m_\circ=\Delta=0,\quad m_1=0,\quad m_2=0,\quad m_3=0,$$

$$m_4=(b_2'-\lambda b_3')[b_5'(\omega_6\omega_{14}-\omega_5\omega_{15})+b_6'(\omega_4\omega_{15}-\omega_6\omega_{13})+b_7'(\omega_5\omega_{13}-\omega_4\omega_{14})]$$

$$m_5=(b_2'-\lambda b_3')[b_4'(\omega_5\omega_{15}-\omega_6\omega_{14})+b_6'(\omega_6\omega_{12}-\omega_3\omega_{15})+b_7'(\omega_3\omega_{14})-\omega_5\omega_{12}]$$

$$m_6=(b_2'-\lambda b_3')[b_4'(\omega_6\omega_{13}-\omega_4\omega_{15})+b_5'(\omega_3\omega_{15}-\omega_6\omega_{12})+b_7'(\omega_4\omega_{12}-\omega_3\omega_{13})]$$

$$m_7=(b_2'-\lambda b_3')[b_4'(\omega_4\omega_{14}-\omega_5\omega_{13})+b_5'(\omega_5\omega_{12}-\omega_3\omega_{14})+b_6'(\omega_3\omega_{13}-\omega_4\omega_{12})]$$

Then the centers of the pseudospheres are an ideal point (point at infinity). The corresponding pseudospheres degenerates into a hyperplane.

Case 5. $rank(B)=2\neq rank(B\setminus H)$. In this case we assume

$$\frac{\omega_3}{\omega_8}=\frac{\omega_4}{\omega_9}=\frac{\omega_5}{\omega_{10}}=\frac{\omega_6}{\omega_{11}}=\lambda,\quad \frac{b_1'}{b_2'}\neq\lambda,$$

$$\frac{\omega_8}{\omega_{12}}=\frac{\omega_9}{\omega_{13}}=\frac{\omega_{10}}{\omega_{14}}=\frac{\omega_{11}}{\omega_{15}}=\mu,\quad \frac{b_2'}{b_3'}\neq\mu.$$

Using the homogenous coordinates

$$m_\circ=\Delta=0,\quad m_1=0,\quad m_2=0,\quad m_3=0$$

$$m_4=(\lambda b_2'-\lambda\mu b_3')[b_5'(\omega_{11}\omega_{14}-\omega_{10}\omega_{15})+b_6'(\omega_9\omega_{15}-\omega_{11}\omega_{13})+b_7'(\omega_{10}\omega_{13}-\omega_9\omega_{14})]$$

$$m_5=(\lambda b_2'-\lambda\mu b_3')[b_4'(\omega_{10}\omega_{15}-\omega_{11}\omega_{14})+b_6'(\omega_{11}\omega_{12}-\omega_8\omega_{15})+b_7'(\omega_8\omega_{14})-\omega_{10}\omega_{12}]$$

$$m_6 = (\lambda b_2' - \lambda\mu b_3')[b_4'(\omega_{11}\omega_{13} - \omega_9\omega_{15}) + b_5'(\omega_8\omega_{15} - \omega_{11}\omega_{12}) + b_7'(\omega_9\omega_{12} - \omega_8\omega_{13})]$$

$$m_7 = (\lambda b_2' - \lambda\mu b_3')[b_4'(\omega_9\omega_{14} - \omega_{10}\omega_{13}) + b_5'(\omega_{10}\omega_{12} - \omega_8\omega_{14}) + b_6'(\omega_8\omega_{13} - \omega_9\omega_{12})]$$

Then we have the same result as in case 4.

Case 6. $rank(B) = 1 \neq rank(B \setminus H)$. In this case the centers of the possible pseudospheres tends to a straight line at infinity. The corresponding pseudospheres degenerate and formed a pencil of hyperplanes. They contain 4 - dimensional subspaces, which contains the given starting pseudohyperbolic surface $k_\circ$ and the corresponding velocity vectors. This leads directly to the well known result in E^3, that there is in general will be no series of pseudospheres tangent to the three-dimensional surfaces.

4. Curve of centers of the pseudospheres

Now, we consider t is varying and in this section, we will determine the centers of pseudospheres which contain a pseudohyperbolic surface $k(t)$ and are tangent to all tangent planes $\tau(t,\theta,\phi)$ of the three-dimensional surface (2). Let $a_i(t), i = 1,2,...,7$ are the column vectors of the matrix $A(t)$, then (2) can be represented in the following way

$$X(t,\theta,\phi) = s(t)[a_1(t)\cosh\theta + a_2(t)\sinh\theta\sin\phi + a_3(t)\sinh\theta\cos\phi] + d(t), \qquad (8)$$

where $d(t)$ is the center of the moving pseudohyperbolic surface and $a_1(t), a_2(t), a_3(t)$ are three orthogonal vectors in the space of the moving pseudohyperbolic surface. The velocity vectors of the points of the sphere are given by

$$\begin{aligned} X'(t,\theta,\phi) = \; & [s'(t)a_1(t) + s(t)a_1'(t)]\cosh\theta + [s'(t)a_2(t) + s(t)a_2'(t)]\sinh\theta\sin\phi \\ & + [s'(t)a_3(t) + s(t)a_3'(t)]\sinh\theta\cos\phi + d'(t). \end{aligned} \qquad (9)$$

The equation of the hyperplanes orthogonal to such a path is

$$Y^T X'(t,\theta,\phi) = X^T(t,\theta,\phi) X'(t,\theta,\phi),$$

where $Y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)^T$ is the position vector of an arbitrary point Y in the hyperplane. The scalar product in the above equation is Lorentz metric. According to the inner product this equation is

$$Y^T \varepsilon X'(t,\theta,\phi) = X^T(t,\theta,\phi)\varepsilon X'(t,\theta,\phi), \qquad (10)$$

where $\varepsilon = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ is the sign matrix. Substitution of equations (8) and (9) into (10), yields

$$
\begin{aligned}
Y^T \quad & \varepsilon[s'(t)a_1(t)+s(t)a_1'(t)]\cosh\theta + Y^T\varepsilon[s'(t)a_2(t)+s(t)a_2'(t)]\sinh\theta\sin\phi \\
& +Y^T\varepsilon[s'(t)a_3(t)+s(t)a_3'(t)]\sinh\theta\cos\phi + Y^T\varepsilon d'(t) \\
& = (s(t)a_1^T(t)\cosh\theta + s(t)a_2^T(t)\sinh\theta\sin\phi + s(t)a_3^T(t)\sinh\theta\cos\phi + d^T(t)) \\
& \varepsilon([s'(t)a_1(t)+s(t)a_1'(t)]\cosh\theta + [s'(t)a_2(t)+s(t)a_2'(t)]\sinh\theta\sin\phi \\
& +[s'(t)a_3(t)+s(t)a_3'(t)]\sinh\theta\cos\phi + d'(t)).
\end{aligned} \tag{11}
$$

Since $A^T\varepsilon A=\varepsilon$ and $A^T\varepsilon A'$ is a skew symmetric matrix, let $e_k(t)=a_k^T(t)\varepsilon d'(t), h_k(t)=a_k'(t)\varepsilon d^T(t)$ and $\ell_k(t)=a_k(t)\varepsilon d^T(t), k=1,2,3$. Then by comparing the coefficients of

$\{1,\cosh\theta,\sinh\theta\sin\phi,\sinh\theta\cos\phi\}$ in (11), we obtain

$$
\begin{aligned}
\sum_{i=1}^{7}\varepsilon_i y_i b_i'(t) &= \sum_{i=1}^{7}\varepsilon_i b_i(t)b_i'(t) - s(t)s'(t), \\
s'(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i1}(t) + s(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i1}'(t) &= s(t)(e_1(t)+h_1(t)) + s'(t)\ell_1(t), \\
s'(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i2}(t) + s(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i2}'(t) &= s(t)(e_2(t)+h_2(t)) + s'(t)\ell_2(t), \\
s'(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i3}(t) + s(t)\sum_{i=1}^{7}\varepsilon_i y_i a_{i3}'(t) &= s(t)(e_3(t)+h_3(t)) + s'(t)\ell_3(t).
\end{aligned} \tag{12}
$$

where $\varepsilon_1=-1$, $\varepsilon_j=1$, $j=2,3,4,5,6,7$. We know from the initial position, that the hyperplanes of the three-dimensional surfaces contain a point $m(t)$ for any t and $\forall\theta,\phi$ such that $m(t)=(0,0,0,m_4(t),m_5(t),m_6(t),m_7(t))$ is the center of this pseudosphere, then from (12), one can find

$$FM = Q, \tag{13}$$

where

$$
F=\begin{pmatrix}
b_4'(t) & b_5'(t) & b_6'(t) & b_7'(t) \\
s'(t)a_{41}+s(t)a_{41}'(t) & s'(t)a_{51}+s(t)a_{51}'(t) & s'(t)a_{61}+s(t)a_{61}'(t) & s'(t)a_{71}+s(t)a_{71}'(t) \\
s'(t)a_{42}+s(t)a_{42}'(t) & s'(t)a_{52}+s(t)a_{52}'(t) & s'(t)a_{62}+s(t)a_{62}'(t) & s'(t)a_{72}+s(t)a_{72}'(t) \\
s'(t)a_{43}+s(t)a_{43}'(t) & s'(t)a_{53}+s(t)a_{53}'(t) & s'(t)a_{63}+s(t)a_{63}'(t) & s'(t)a_{73}+s(t)a_{73}'(t)
\end{pmatrix},
$$

$$
M=\begin{pmatrix} m_4(t) \\ m_5(t) \\ m_6(t) \\ m_7(t) \end{pmatrix} \text{ and } Q=\begin{pmatrix} \sum_{i=1}^{7}\varepsilon_i b_i(t)b_i'(t) - s(t)s'(t) \\ s(t)(e_1(t)+h_1(t)) + s'(t)\ell_1(t) \\ s(t)(e_2(t)+h_2(t)) + s'(t)\ell_2(t) \\ s(t)(e_3(t)+h_3(t)) + s'(t)\ell_3(t) \end{pmatrix}.
$$

If F is a regular matrix, we get

$$M = F^{-1}Q. \tag{14}$$

Therefor, the coordinates of the centers of the pseudospheres in the fixed frame at any instant t are given by

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ \end{pmatrix} = s(t)A(t)\begin{pmatrix} 0 \\ 0 \\ 0 \\ m_4(t) \\ m_5(t) \\ m_6(t) \\ m_7(t) \\ \end{pmatrix} + d(t). \tag{15}$$

Theorem 4.1

Example 1

References

[1] Abdel-All, N. H. and Hamdoon, F. M. 2004. Cyclic surfaces in E^5 generated by equiform motions, J. Geom. **79** (2004) 1–11.
[2] Bottema, O. and Roth, B. 1990. Theoretical kinematic, Dover Publications Inc., New York, 1990.
[3] Gordon, V. O. and Sement Sov, M. A. 1980. A Course in Descriptive Geometry. Mir Publishers, Moscow, 1980.
[4] Jagy, W. 1998. Sphere foliated constant mean curvature submanifolds, Rocky Mount. J. Math., **28** (1998) 983–1015.
[5] López, R. 1999. constant mean curvature hypersrfaces foliated by spheres, J. of Differential Geometry and its applications. **11** (1999), 245–256.
[6] López, R. 2001. How to use MATHEMATICA to find cyclic surfaces of constant curvature in Lorentz-Minkowski space, in: Global Differential Geometry: The Mathematical Legacy of Alfred Gray, (M. Fernández, J. Wolf, Ed.) Contemporary Mathematics, 288, A. M. S., 2001, 371–375.
[7] López, R. 2000. Maximal surfaces of Riemann type in Lorentz-Minkowski space $\mathbb{L}^3$, Michigan Math. J.. **47** (2000), 469–496.
[8] Nassar H. Abdel-All, Hamdoon, F. M. 2004. A geomertic characterisation of two-parameter spatial motions with many locally one-dimensional point paths, J. of Applied Math. and computation, **153** (2004) 19–25.
[9] Odehnal, B., Pottmann, H. and Wallner, J. 2006. Equiform kinematics and the geometry of line elements, Beitr. Algebra Geom., **47** (2006), 567–582.
[10] O'Neill, B. 1983. Semi-Riemannian Geometry with Application to Relativity, Academic Press, (1983).
[11] Pottmann, H. and Wallner, J. 2001. Computational Line Geometry, Springer Verlag, Heidelberg, Berlin 2001.
[12] Solouma, E. M. 2012. Local study of scalar curvature of two-dimensional surfaces obtained by the motion of circle, J. of Applied Math. and computation **219** (2012) 3385–3394.